Large-eddy simulations of compressible MHD turbulence

Arakel Petrosyan
Theoretical section,
Space Research Institute of the Russian Academy of Science,
Moscow, Russia

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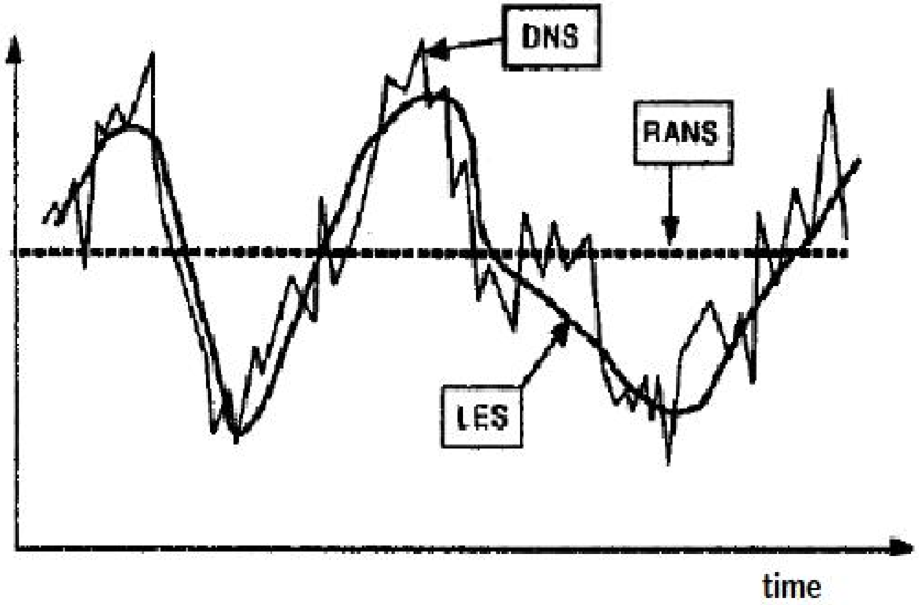
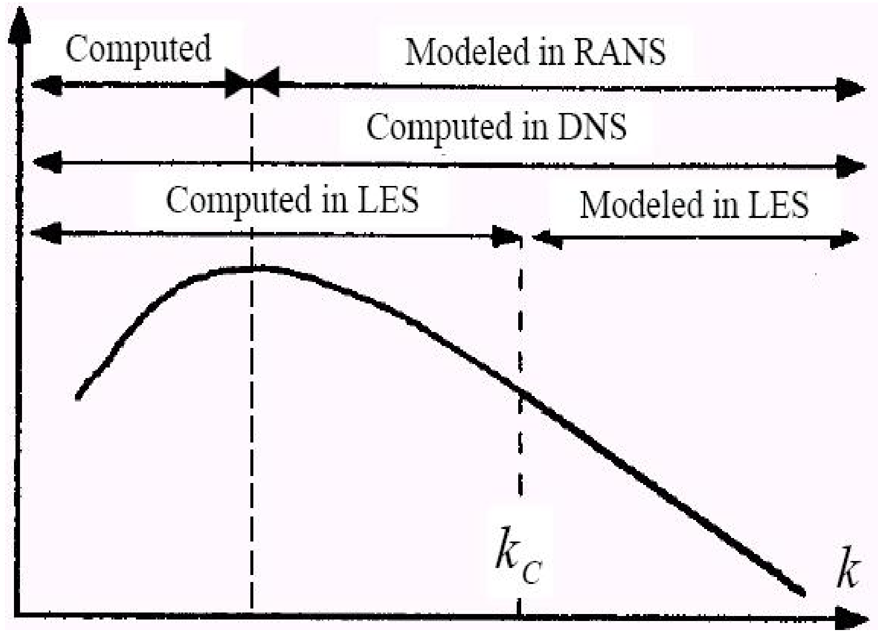
RANS, DNS & LES





$$u_i = \overline{u_i} + u_i',$$

$$\bar{\bar{a}} = \bar{a},$$

$$\overline{a+b} = \bar{a} + \bar{b},$$

$$\overline{a\bar{b}} = \bar{a}\bar{b}.$$

$$R_{ij} \equiv \overline{\rho u_i' u_j'}$$

$$f = \widetilde{f} + f''$$

$$\overline{f}(x) = \int_D f(x')G(x,x';\overline{\Delta})dx'$$

Properties:

$$\overline{\rho u''} \neq 0$$

$$\widetilde{\widetilde{u}} \neq \widetilde{u}$$

$$\widetilde{\widetilde{u}} \neq 0$$

MHD equations

$$
\begin{cases}
\dfrac{\partial \rho}{\partial t} = -\dfrac{\partial \rho u_j}{\partial x_j} \\[2ex]
\dfrac{\partial \rho u_i}{\partial t} = -\dfrac{\partial}{\partial x_j}\left(\rho u_i u_j + p\delta_{ij} - \sigma_{ij} - \dfrac{1}{4\pi} B_j B_i + \dfrac{1}{8\pi} B^2\right) \\[2ex]
\dfrac{\partial B_i}{\partial t} = -\dfrac{\partial}{\partial x_j}\left(B_i u_j - B_j u_i\right) + \eta \nabla^2 B_i \\[2ex]
\dfrac{\partial B_j}{\partial x_j} = 0
\end{cases}
$$

Polytropic relation: $p = \rho^{\gamma}$

Mass weighted filtering

$$\frac{\partial \bar{\rho}\bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{\rho}\bar{u}_i\bar{u}_j + \bar{p}\delta_{ij} - \bar{\sigma}_{ij} + \frac{\bar{B}^2}{8\pi}\delta_{ij} - \frac{1}{4\pi}\,\bar{B}_j\bar{B}_i\right) = -\frac{\partial}{\partial x_j}\left(\overline{\rho u_j u_i} - \bar{\rho}\bar{u}_i\bar{u}_j\right) +$$

$$+\frac{1}{4\pi}\frac{\partial}{\partial x_j}\left(\overline{B_j B_i} - \bar{B}_i\bar{B}_j\right) - \frac{\partial}{\partial t}\left(\overline{\rho u_i} - \bar{\rho}\bar{u}_i\right)$$

additional terms, which are necessary to parametrize!

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \bar{\rho}\bar{u}_j}{\partial x_j} = -\frac{\partial}{\partial x_j}\left(\overline{\rho u_j} - \bar{\rho}\bar{u}_j\right)$$

$$\tilde{f} = \frac{\overline{\rho f}}{\bar{\rho}}$$

$$\tilde{u}_j = \frac{\overline{\rho u_j}}{\bar{\rho}} = \frac{\int_a^b \rho u_j G(x_j - x'_j, \bar{\Delta}_j)\, dx'_j}{\int_a^b \rho(x'_j) G(x_j - x'_j, \bar{\Delta}_j)\, dx'_j}.$$

Filtered MHD equations

$$
\left\{
\begin{aligned}
&\frac{\partial \bar\rho}{\partial t} + \frac{\partial \bar\rho \tilde u_j}{\partial x_j} = 0 \\
&\frac{\partial \bar\rho \tilde u_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar\rho \tilde u_i \tilde u_j + \bar p \delta_{ij} - \frac{1}{\mathrm{Re}} \tilde\tau_{ij} + \frac{\overline{B}^2}{2M_A^2} - \frac{1}{M_A^2} \overline{B}_i \overline{B}_j\right) = -\frac{\partial \tau^u_{ji}}{\partial x_j} \\
&\frac{\partial \overline{B}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\tilde u_j \overline{B}_i - \overline{B}_j \tilde u_i\right) - \frac{1}{\mathrm{Re}_m} \frac{\partial^2 \overline{B}_i}{\partial x_j^2} = -\frac{\partial \tau^b_{ji}}{\partial x_j}
\end{aligned}
\right.
$$

Subgrid -scale (SGS) or
Subfilter-scale (SFS) terms

Subgrid-scale modeling -1

Eddy-viscosity model:

$$\tau^{u}_{ij} - \frac{1}{3}\tau^{u}_{kk}\delta_{ij} = -2\nu_{t}(\tilde{S}_{ij} - \frac{1}{3}\tilde{S}_{kk}\delta_{ij})$$

$$\tilde{S}_{ij} = \frac{1}{2}(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i})$$ - large-scale strain rate tensor

$$\tau^{b}_{ij} - \frac{1}{3}\tau^{b}_{kk}\delta_{ij} = -2\eta_{t}\bar{J}_{ij}$$

$$\bar{J}_{ij} = \frac{1}{2}(\frac{\partial \bar{B}_i}{\partial x_j} - \frac{\partial \bar{B}_j}{\partial x_i})$$ - large-scale magnetic rotation tensor

$$\nu_t = C\phi \qquad \eta_t = D\gamma$$

Following Prandtl's mixing length phenomenology turbulent viscosity and magnetic diffusion can both be approximated as the product of the grid filter width and a characteristic velocity V or magnetic field strength B. Since V and *B* are hard to determine directly in a homogeneously turbulent flow, one can instead extract estimates for them from the respective subgrid energy-dissipation.

$$\phi \sim \overline{l}^{4/3}(\varepsilon^{K})^{1/3}$$

$$\gamma \sim \overline{l}^{4/3}(\varepsilon^{M})^{1/3}$$

dimensional considerations

ε^{K} - kinetic subgrid-energy dissipation

ε^{M} - magnetic subgrid-energy dissipation

The subgrid models differ in the way they estimate these dissipation functions

Smagorinsky model for MHD

Approximating the subgrid energy dissipation with the aid of the local resolved dissipaion leads to the classical moc $\varepsilon^K \sim l^2(2\bar{\mathbf{S}}^v:\bar{\mathbf{S}}^v)^{3/2}$ $\quad \varepsilon^M \sim l^2|\bar{\mathbf{j}}|^3$

$$\tau^u_{ij} - \frac{1}{3}\tau^u_{kk}\delta_{ij} = -2\nu_t(\tilde{S}_{ij} - \frac{1}{3}\tilde{S}_{kk}\delta_{ij}) \qquad \tau^u_{kk} = 2Y_1\bar{\rho}\bar{\Delta}^2|\tilde{S}^u|^2 \qquad |\tilde{S}^u| = (2S_{ij}S_{ij})^{1/2}$$

Turbulent viscosity: $\nu_t = C_1\bar{\rho}\bar{\Delta}^2|\tilde{S}^u|$

$$\tilde{S}_{ij} = \frac{1}{2}\left(\frac{\partial\tilde{u}_i}{\partial x_j} + \frac{\partial\tilde{u}_j}{\partial x_i}\right)$$ - large-scale strain rate tensor

$$\bar{J}_{ij} = \frac{1}{2}\left(\frac{\partial\bar{B}_i}{\partial x_j} - \frac{\partial\bar{B}_j}{\partial x_i}\right)$$ - large-scale magnetic rotation tensor

$$\tau^b_{ij} - \frac{1}{3}\tau^b_{kk}\delta_{ij} = -2\eta_t\bar{J}_{ij}$$

Turbulent diffusivity: $\eta_t = D_1\bar{\Delta}^2|j|$

Kolmogorov model for MHD

If the grid filter cutoff lies within the inertial spectral range of the homogeneously turbulent system and the nonlinear exchange between resolved kinetic and magnetic energy is much smaller than the respective dissipation, kinetic subgrid-energy dissipation and magnetic subgrid-energy dissipation can be assumed to depend only on time. Thus except a unit factor carrying the necessary dimensions and an explicit filter scale dependence, both functions and can be absorbed by the nondimensional parameters yielding the Kolmogorov scaling model

$$\nu_t = C_2 \varepsilon \overline{\Delta}^{4/3}$$ - *turbulent viscosity*

$$\tau^u_{kk} = 2Y_2 \varepsilon \overline{\Delta}^{4/3} | \widetilde{S}^u |$$ - *isotropic term*

$$\eta_t = D_2 \overline{\Delta}^{4/3}$$ - *turbulent magnetic diffusivity*

Cross-helicity model

The cross helicity is $H^c = \int_V (u \cdot B) dV$

With regard to the mixing length framework outlined above the functions ϕ and γ are estimated as the product of subgrid dissipation and an associated length scale. However, instead of the local resolved kinetic and magnetic energy dissipation terms, the corresponding local cross-helicity dissipation expressions

$$\overline{\varepsilon}^{Cv} \sim \overline{\mathbf{S}}^v : \overline{\mathbf{S}}^b \qquad \overline{\varepsilon}^{Cb} \sim \overline{\mathbf{j}} \cdot \overline{\boldsymbol{\omega}}$$

the resolved vorticity $\overline{\boldsymbol{\omega}} = \boldsymbol{\nabla} \times \overline{\mathbf{v}}$

the electric current density $\mathbf{j} = \boldsymbol{\nabla} \times \mathbf{b}$

The cross-helicity is related to the transfer between kinetic and magnetic energies caused by the Lorentz force. Therefore, the cross helicity allows one to estimate the energy exchange between large and small scales in the LES method:

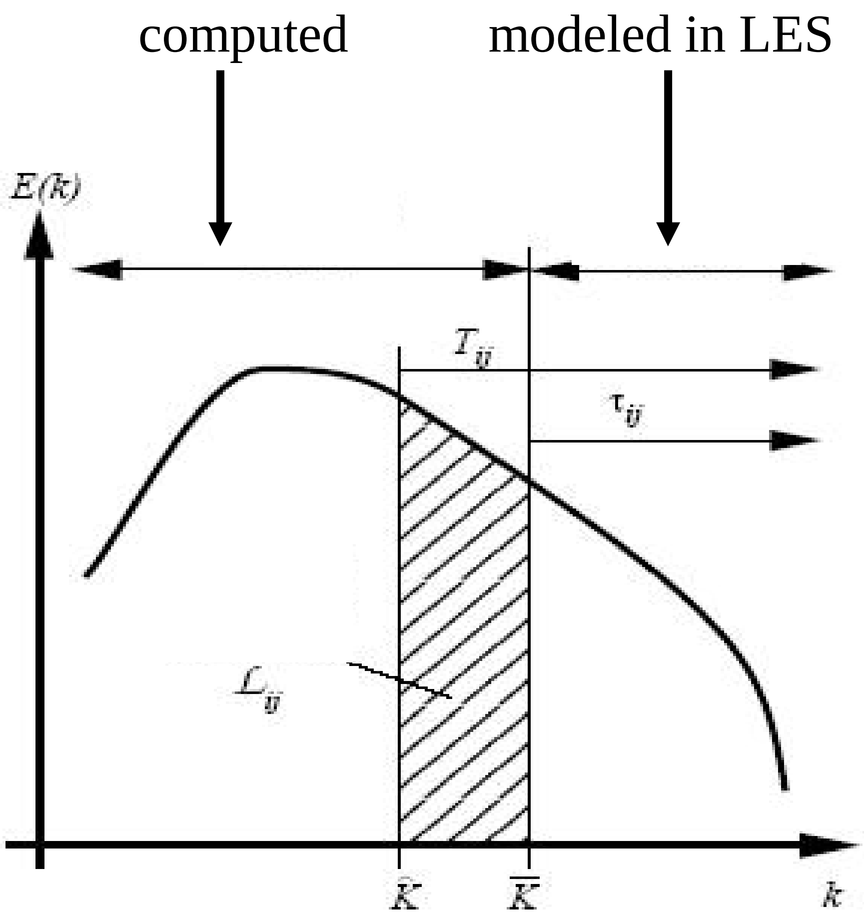
$$\neg_t = C_3 \overline{\circ} \overline{\Delta}^2 \, | \tilde{S}_{ij}^u \tilde{S}_{ij}^b |^{1/2}$$ - *turbulent viscosity*

$$2_{kk}^u = 2Y_3 \overline{\circ} \overline{\Delta}^2 \, | \tilde{f} \, || \tilde{S}^u |$$ - *isotropic term*

$$\bar{f} = | \tilde{S}_{ij}^u \overline{S}_{ij}^b |^{1/2}$$

$$¦_t = D_3 \overline{\Delta}^2 \operatorname{sgn}(\bar{j} \tilde{\mu}) \, | \bar{j} \tilde{\mu} |^{1/2}$$ - *turbulent magnetic diffusivity*

Dynamic procedure



$$T_{ij}^b = \overline{\tau_{ij}^b} + L_{ij}^b \qquad T_{ij}^u = \widetilde{\tau_{ij}^u} + L_{ij}^u$$

Germano assumed that there is an algebraic relation between the stresses at two different filter levels and the resolved stresses. In this model, the model parameter is determined dynamically, and not an *ad hoc* constant.

Applying a second filter to the filtered momentum equations a similar expression is achieved for the new subtest scale stress tensor Tij.

Dynamic procedure

$\nu_t = C_s \alpha^u_{ij}$ (for τ^u_{ij}). - common form

$$L^u_{ij} = \left(\frac{\widehat{\overline{\rho u_i}\,\overline{\rho u_j}}}{\bar{\rho}}\right) - \frac{\widehat{\overline{\rho u_i}}\,\widehat{\overline{\rho u_j}}}{\hat{\bar{\rho}}} - \frac{1}{M_a^2}\left(\widehat{\bar{B}_i \bar{B}_j} - \hat{\bar{B}}_i\,\hat{\bar{B}}_j\right)$$

The angular brackets indicate spatial averaging

$$Z^u = \langle (T^u_{ij} - \tilde{\tau}^u_{ij} - L^u_{ij})^2 \rangle$$

Least square technique

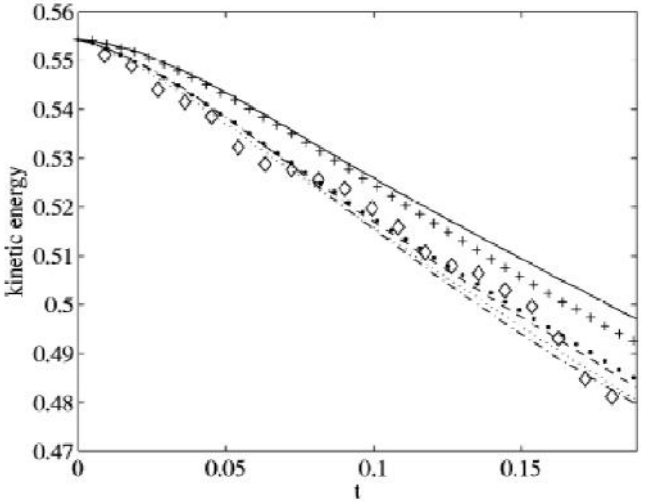
$$C_s = \frac{\langle L^u_{ij}\, M^u_{ij} \rangle}{\langle M^u_{ij}\, M^u_{ij} \rangle}$$

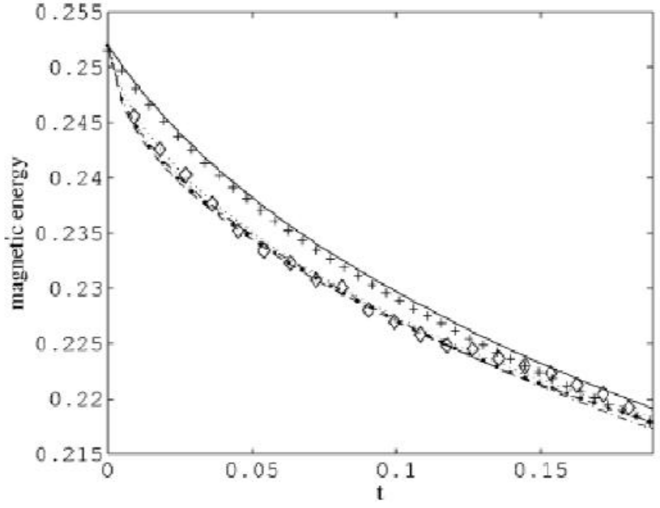
Negative value corresponds to the backscatter energy cascade.

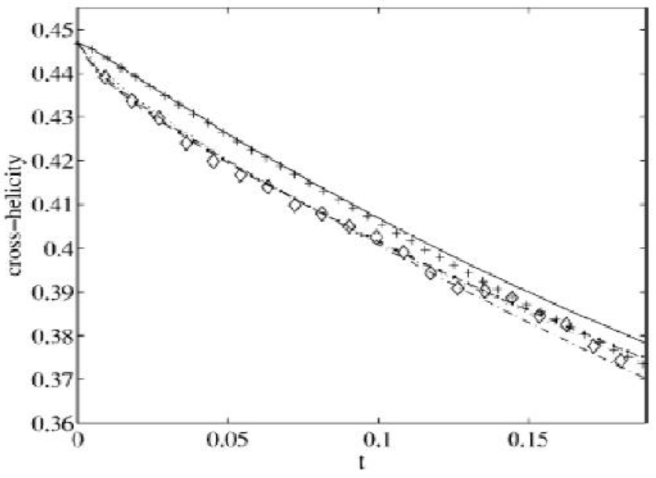
$$M^u_{ij} = \hat{\alpha}^u_{ij}\left(\hat{\tilde{S}}^u_{ij} - \frac{\delta_{ij}}{3}\hat{\tilde{S}}_{kk}\right) - \widehat{\alpha^u_{ij}\left(\tilde{S}^u_{ij} - \frac{\delta_{ij}}{3}\tilde{S}_{kk}\right)}$$

Validation results

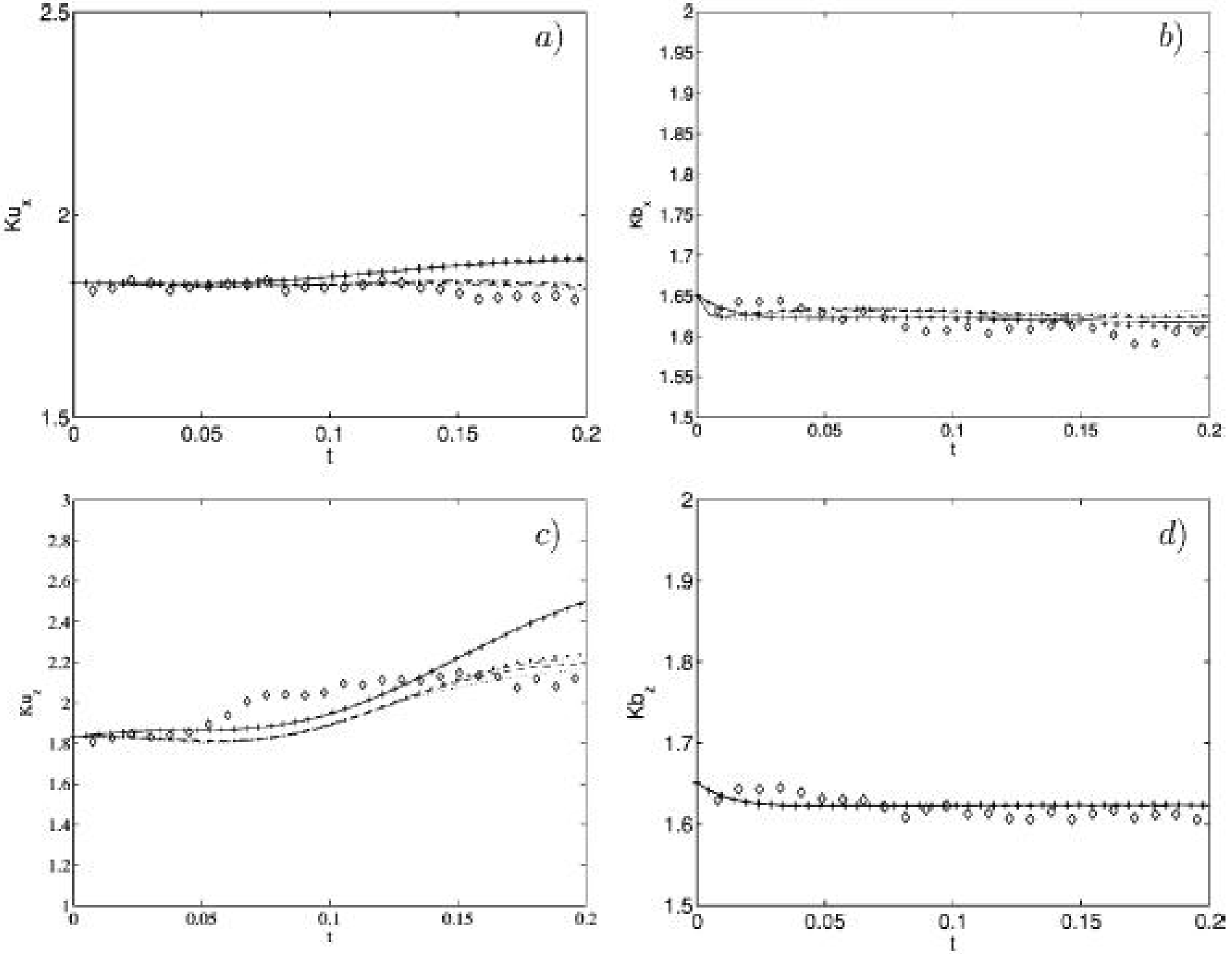
SGS model	Curve
No model	Solid
Smagorinsky model	Dashed
Kolmogorov model	Dotted
Cross-helicity model	Black point
Scale-similarity model	Marker +
Mixed model	Dashed-dotted







Validation results



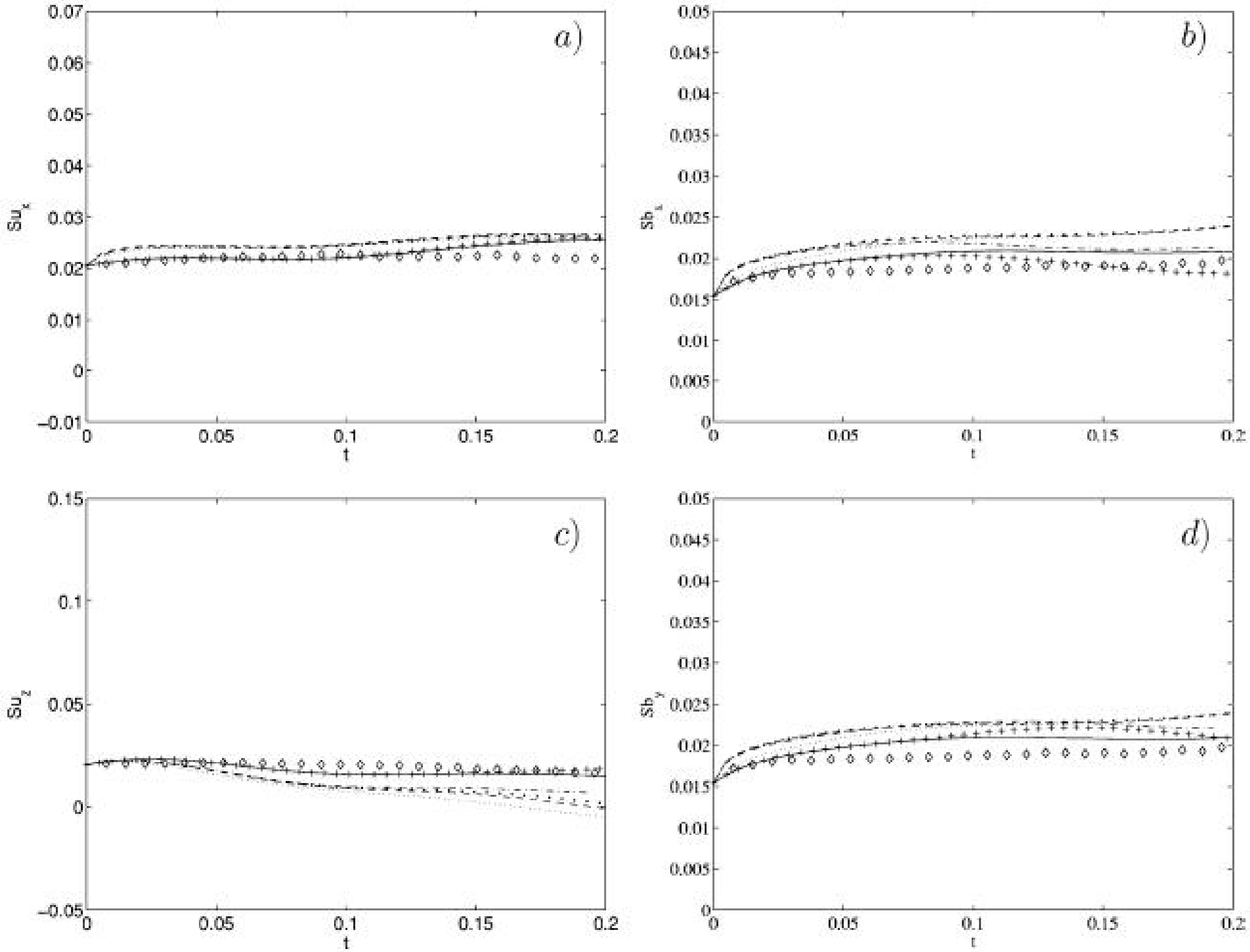
Kurtosis (or flatness) of a velocity component:

$$Ku_j = \frac{\langle u_j^4 \rangle}{(\langle u_j^2 \rangle)^2}$$

Kurtosis (or flatness) of a magnetic field:

$$Kb_j = \frac{\langle B_j^4 \rangle}{(\langle B_j^2 \rangle)^2}$$

Validation results



Skewness of a velocity component:

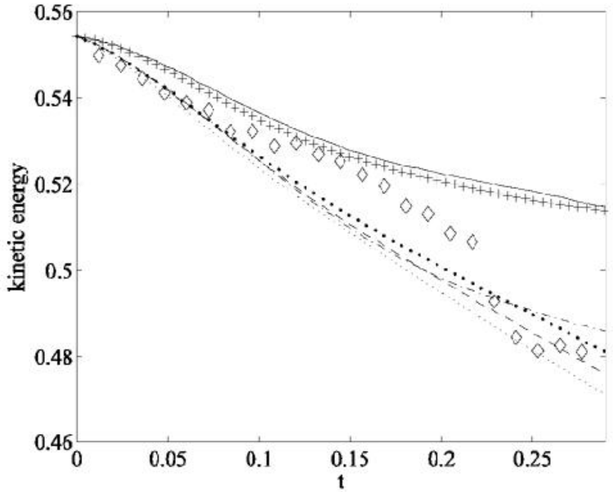
$$Su_j = \frac{\langle u_j^3 \rangle}{(\langle u_j^2 \rangle)^{3/2}}$$

Skewness of a magnetic field:

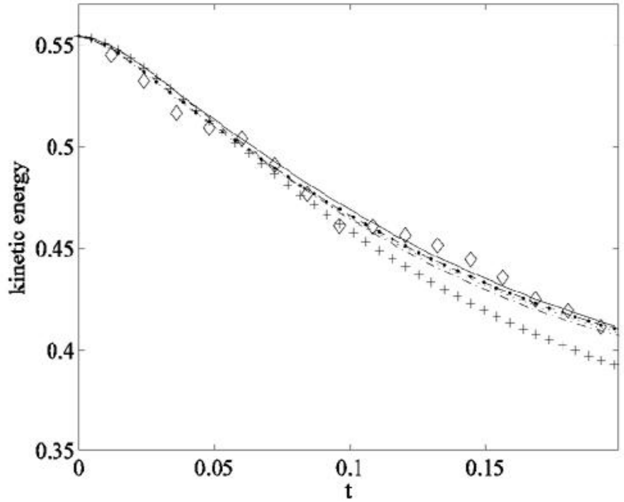
$$Sb_j = \frac{\langle B_j^3 \rangle}{(\langle B_j^2 \rangle)^{3/2}}$$

Kinetic energy

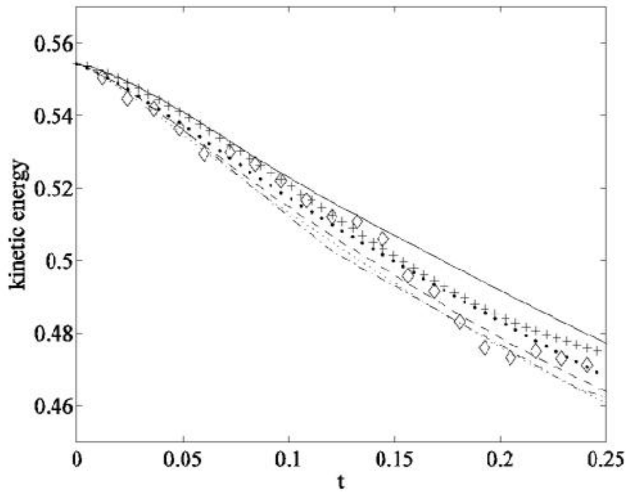
Ms=1



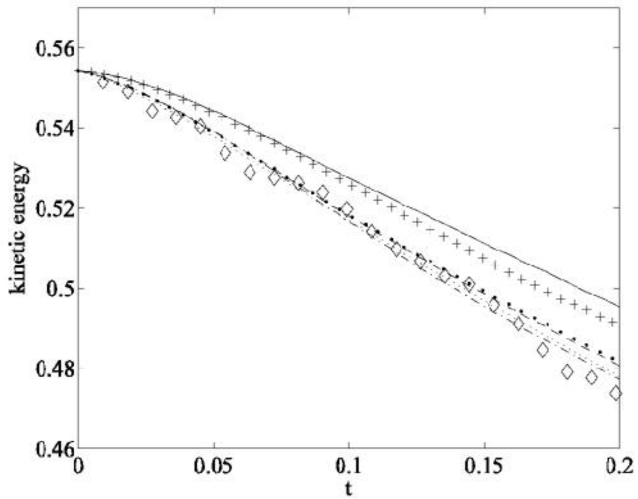
Ms=0.2



Re_m=2

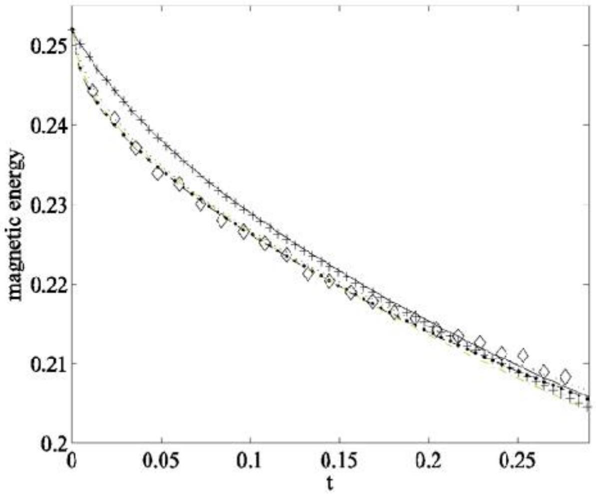


Re_m=20

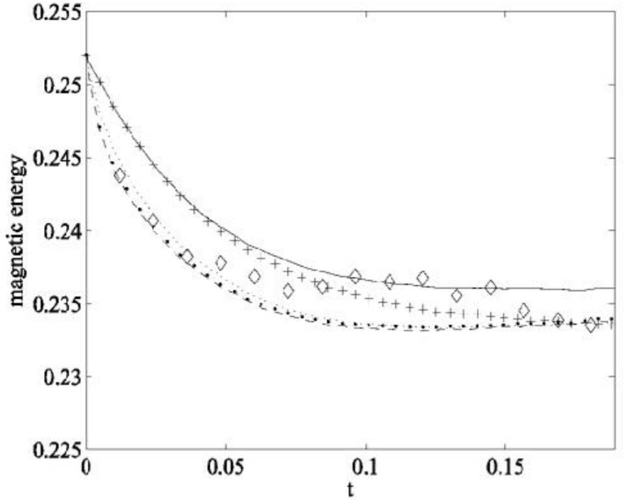


Magnetic energy

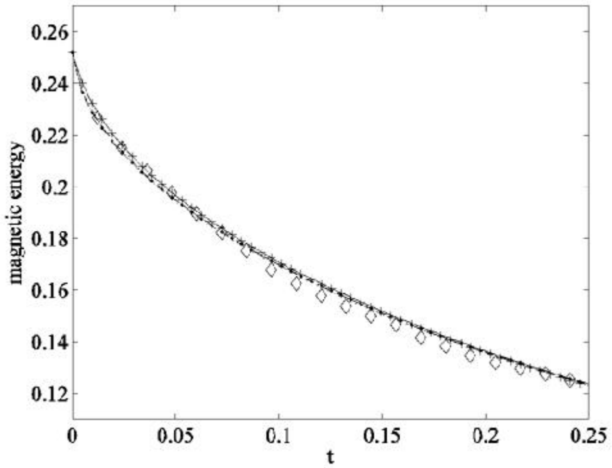
Ms=1



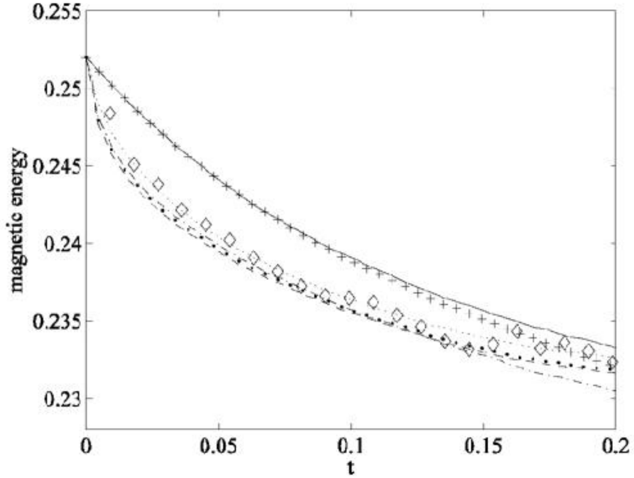
Ms=0.2



Re_m=2

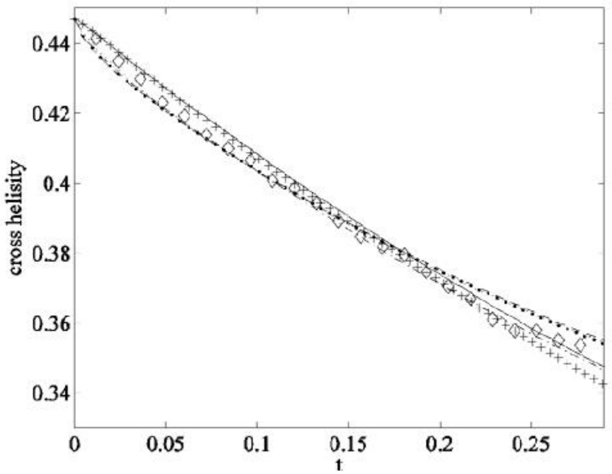


Re_m=20

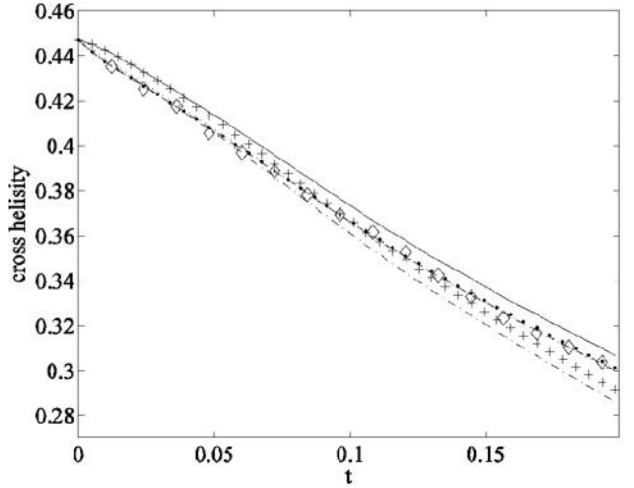


Cross-helicity

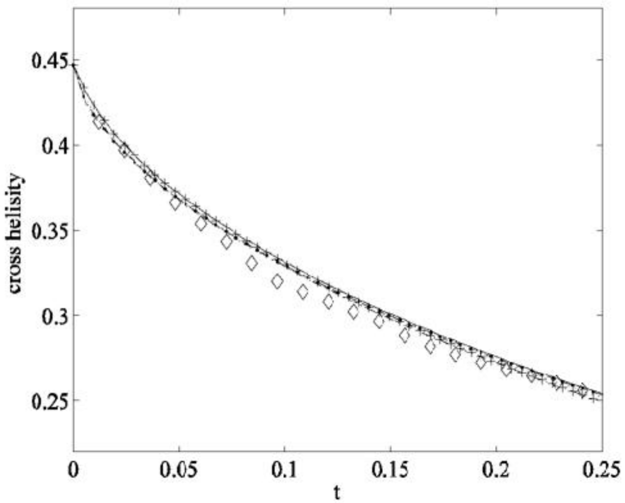
Ms=1



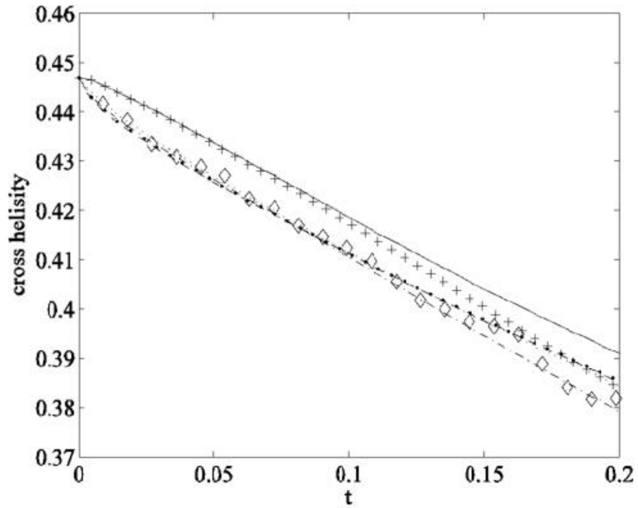
Ms=0.2



Re_m=2



Re_m=20



Skewness and flatness

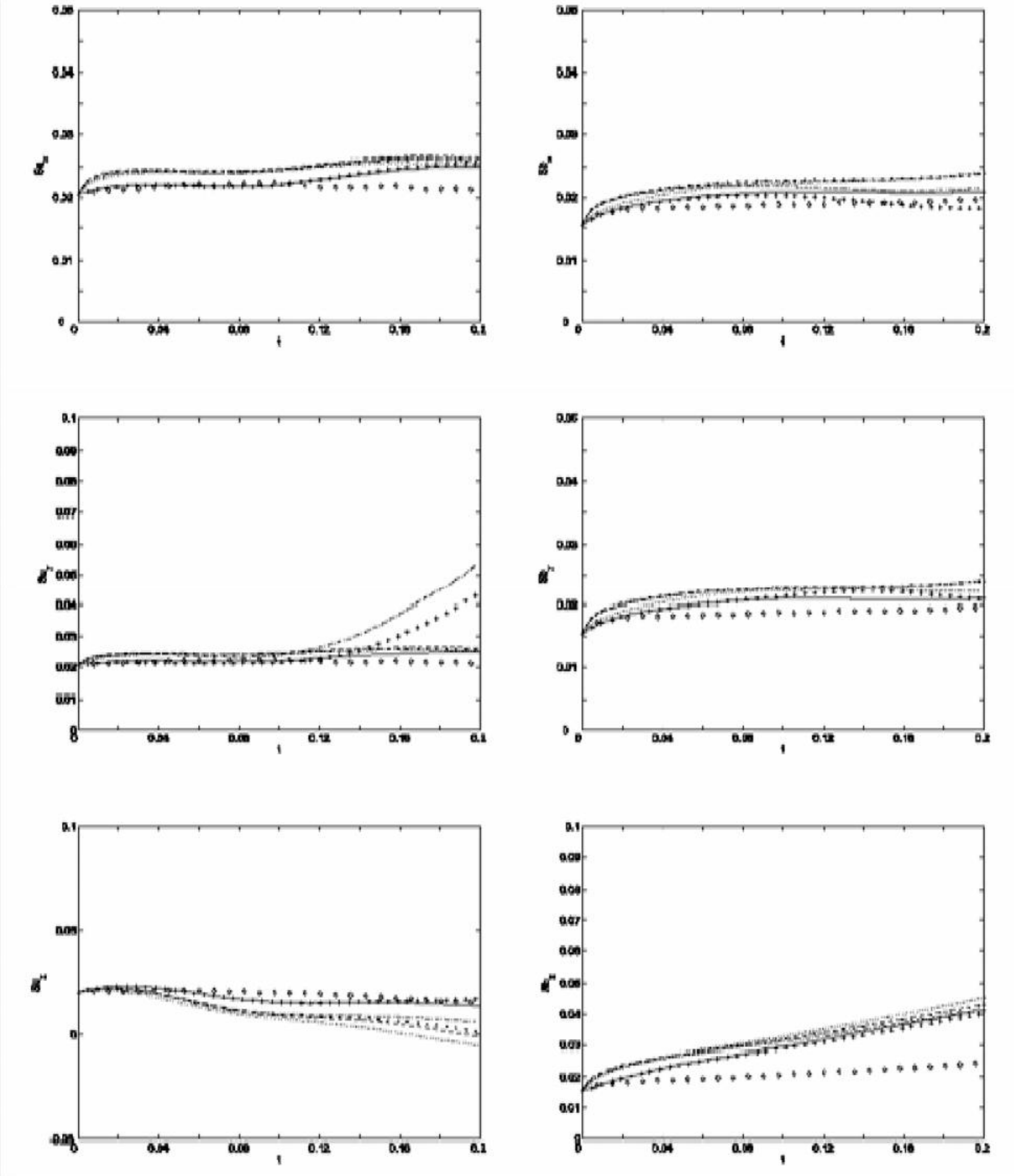
The departure from Gaussianity for fluid turbulence in the laboratory or in numerical simulations is measured in terms of the skewness and flatness factors.

The flatness factor (sometimes also called kurtosis) in turbulent flows is a measure of intermittency. The flatness is an indication of the occurrence of fluctuations far from the mean: it is an indicator of the relative frequency of rare events. Hence the flatness increases with increasing sparseness of the fluctuations:

$$Ku_j = \frac{< u_j^4 >}{(< u_j^2 >)^2} \qquad Kb_j = \frac{< B_j^4 >}{(< B_j^2 >)^2}$$

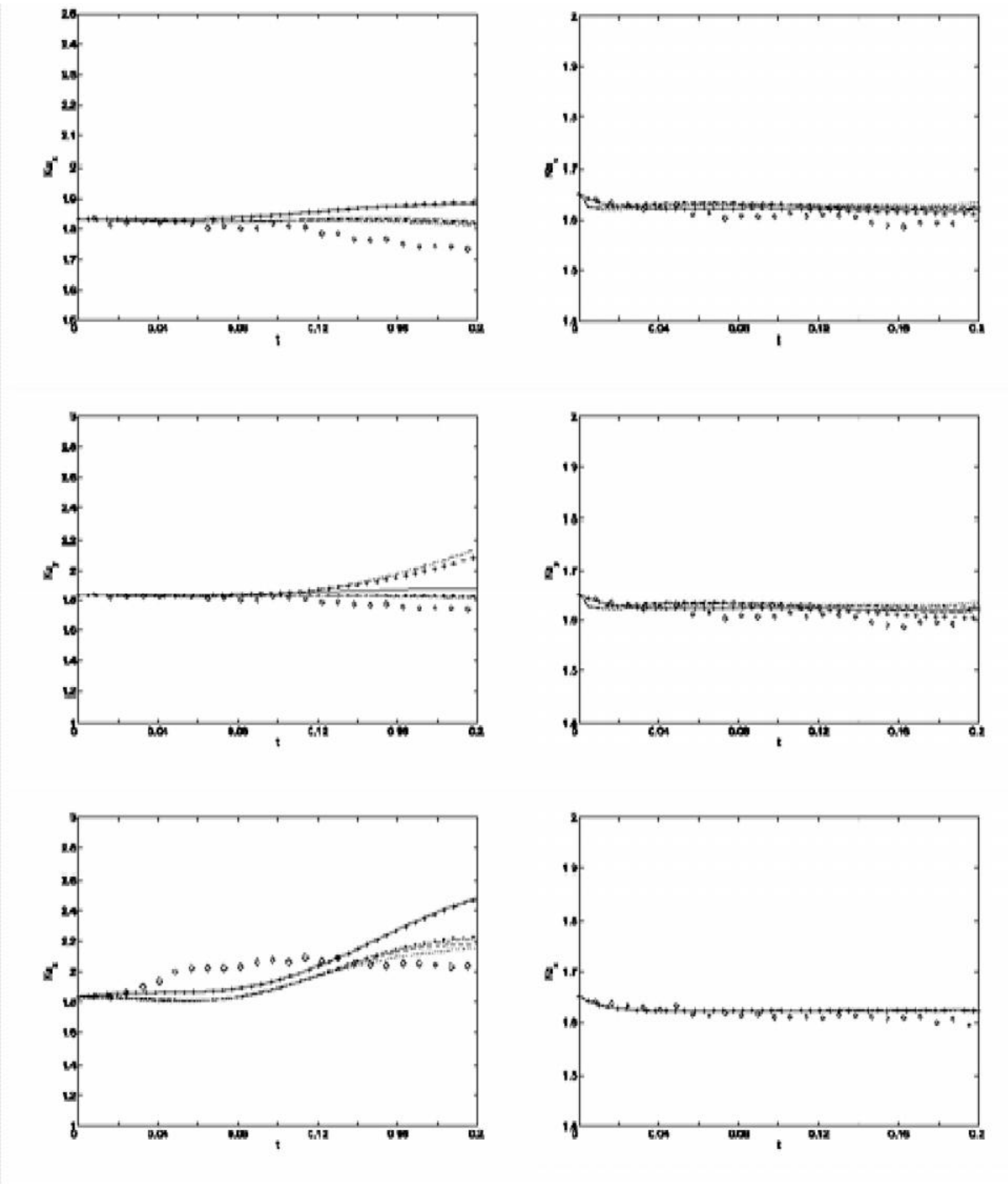
The skewness is related to the asymmetry of the probability density function of the velocity or magnetic filed fluctuations. It is a sensitive indicator of changes in the large scale structure.

$$Su_j = \frac{< u_j^3 >}{(< u_j^2 >)^{3/2}} \qquad Sb_j = \frac{< B_j^3 >}{(< B_j^2 >)^{3/2}}$$



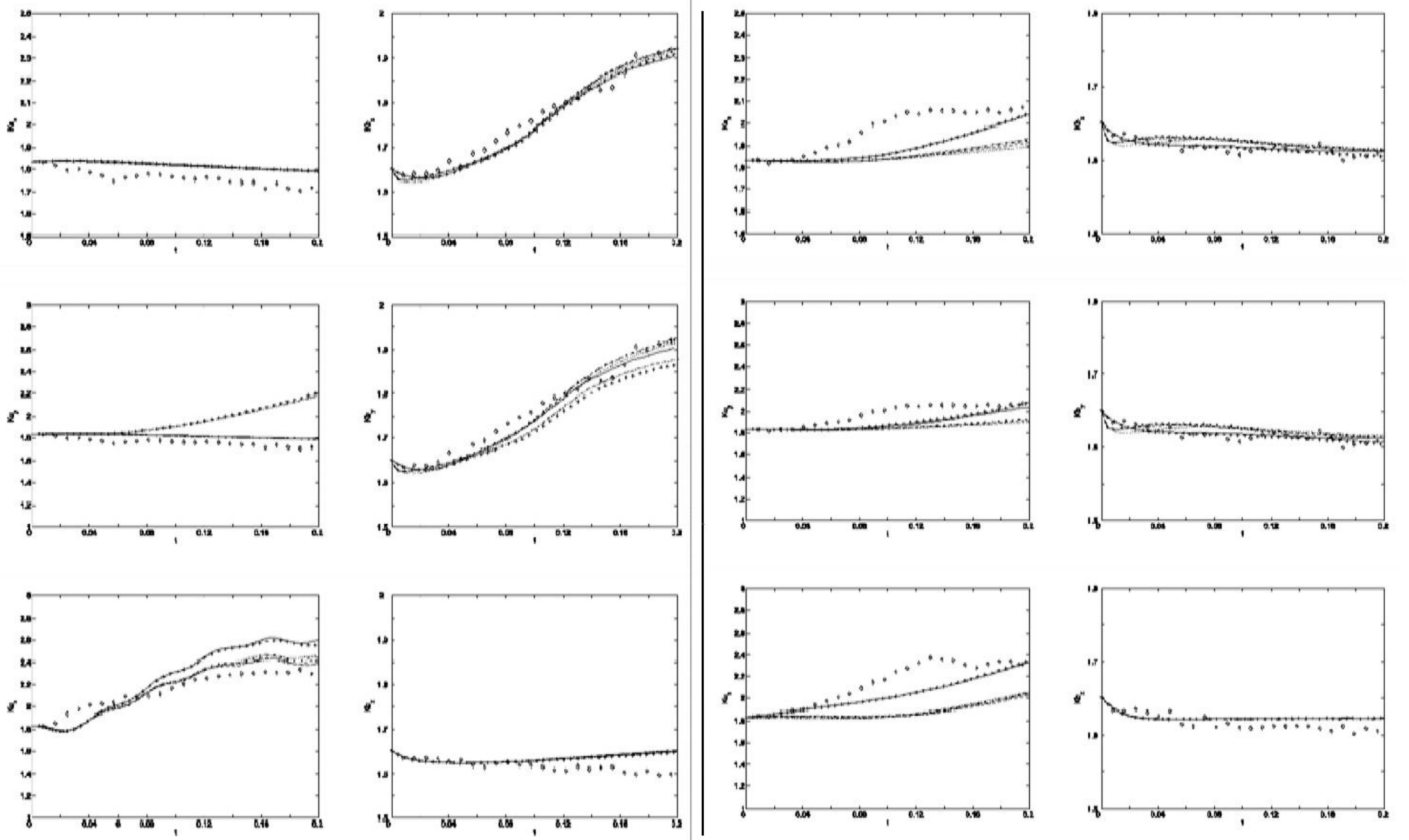
Results	Curve
DNS	Diamond
No model	Solid
Smagorinsky model	Dashed
Kolmogorov model	Dotted
Cross-helicity model	Black point
Scale-similarity model	Marker+
Mixed model	Dashed-dot

Time evolution of skewness and flatness of velocity and magnetic field components for the case Re = 100, Re_l = 25, Re_m = 10.0, M_s = 0.6.



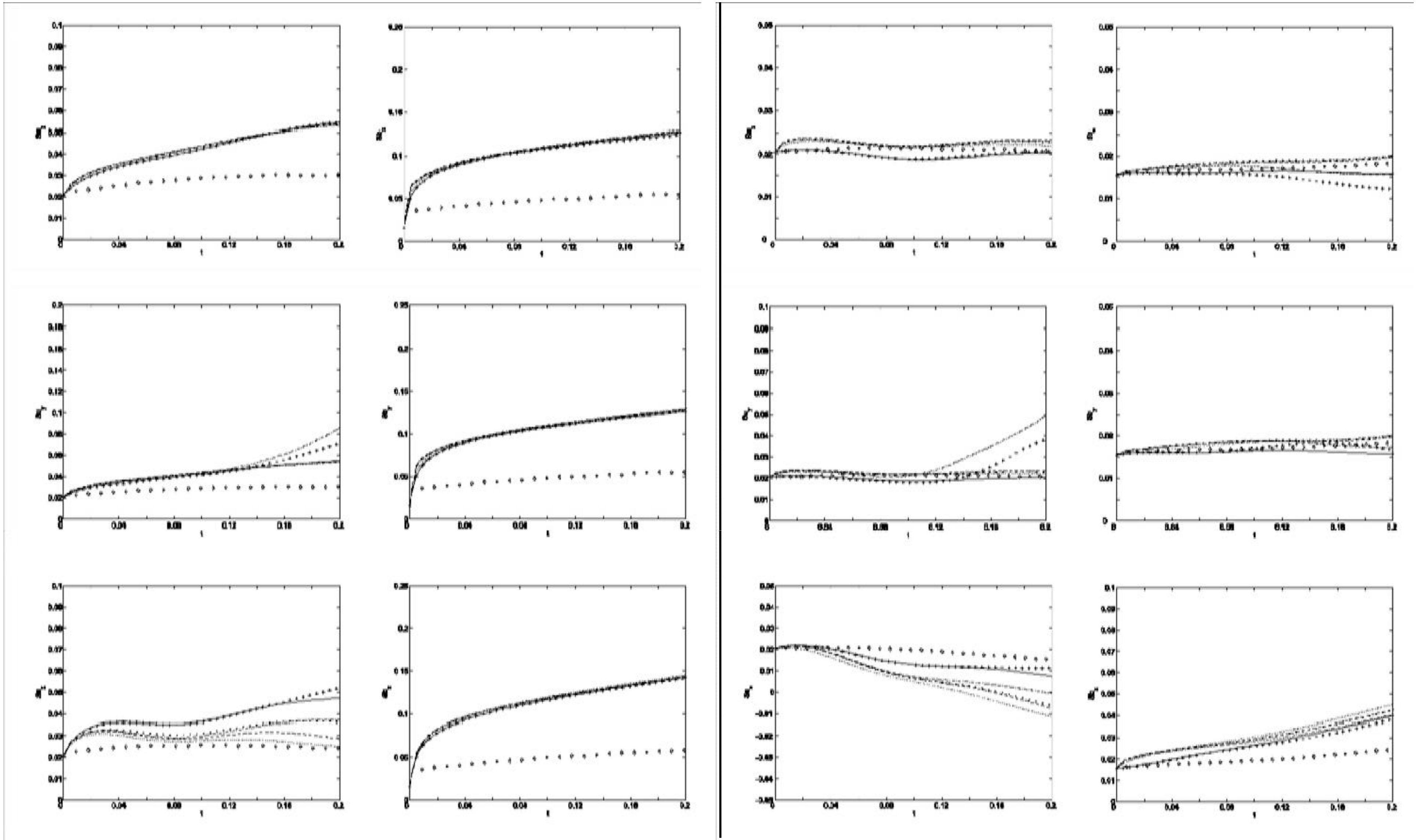
Time dynamics of skewness and flatness of velocity and magnetic field components for the case

$Re = 100$, $Re_l = 25$,
$Re_m = 10.0$, $M_s = 0.6$.



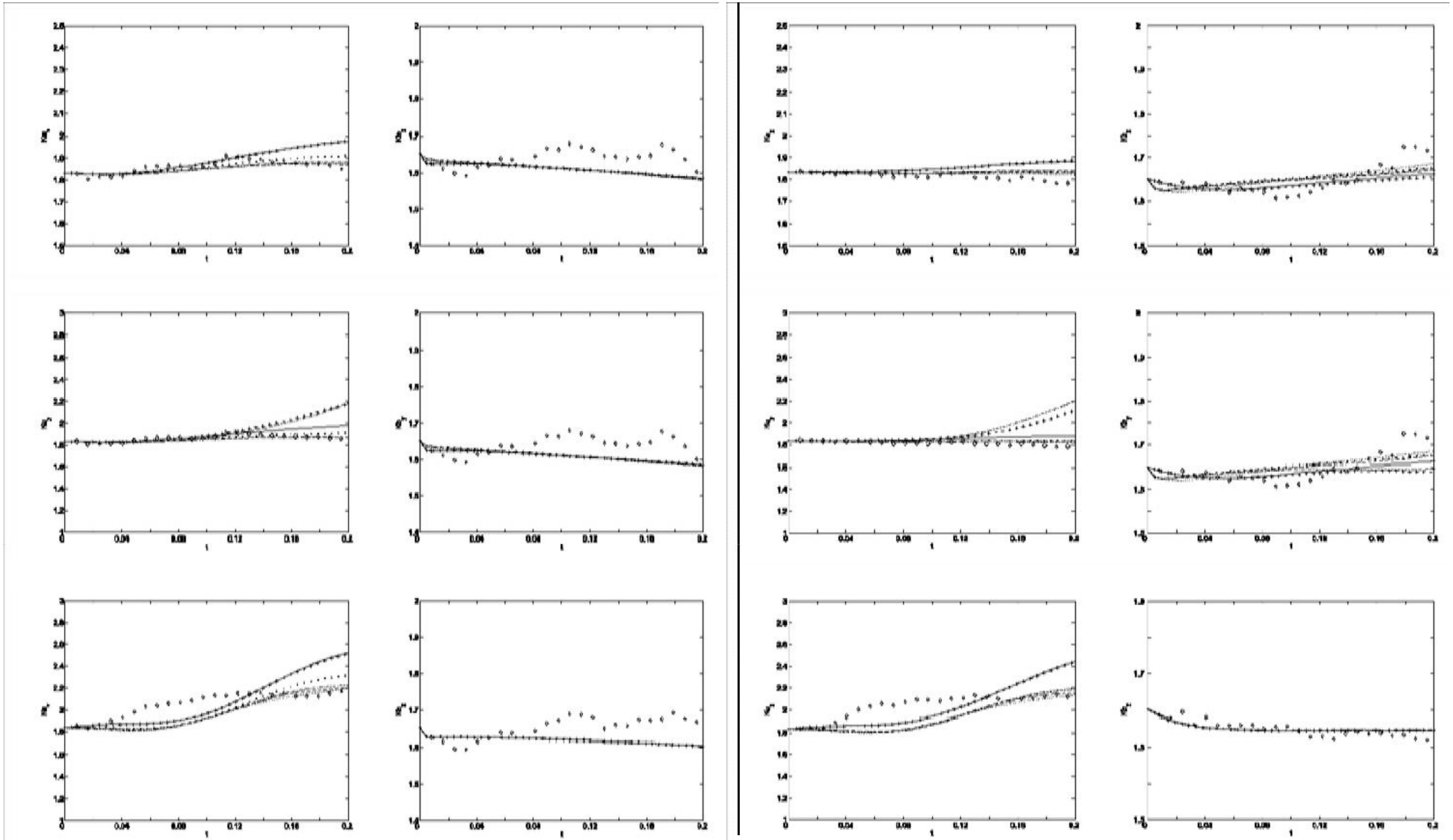
Time dynamics of flatness of velocity and magnetic field components for the case
Re = 390, Re_l = 50, Re_m = 10.0,

M_s = 0.2 **M_s = 1.0.**



Time evolution of skewness of velocity and magnetic field components for the case
Re = 390, Re_l = 50, M_s = 0.6

Re_m = 2.0 **Re_m = 20.0**



Time evolution of flatness of velocity and magnetic field components for the case

Re = 390, Re_l = 50, M_s = 0.6

Re_m = 2.0 **Re_m = 20.0**

Filtered MHD equations for heat-conducting plasma

$$
\begin{cases}
\dfrac{\partial \bar{\circ}}{\partial t} + \dfrac{\partial \bar{\circ}\tilde{u}_j}{\partial x_j} = 0 \\[2ex]
\dfrac{\partial \bar{\circ}\tilde{u}_i}{\partial t} + \dfrac{\partial}{\partial x_j}\left(\bar{\circ}\tilde{u}_i\tilde{u}_j + \bar{p}¢_{ij} - \dfrac{1}{\mathrm{Re}}\tilde{±}_{ij} + \dfrac{\bar{B}^2}{2M_A^2} - \dfrac{1}{M_A^2}\bar{B}_i\bar{B}_j\right) = -\dfrac{\partial 2^{u}_{ji}}{\partial x_j} \\[2ex]
\dfrac{\partial \bar{B}_i}{\partial t} + \dfrac{\partial}{\partial x_j}(\tilde{u}_j\bar{B}_i - \bar{B}_j\tilde{u}_i) - \dfrac{1}{\mathrm{Re}_m}\dfrac{\partial^2 \bar{B}_i}{\partial x_j^2} = -\dfrac{\partial 2^{b}_{ji}}{\partial x_j} \\[2ex]
\dfrac{\partial \bar{\circ}\tilde{E}}{\partial t} + \dfrac{\partial}{\partial x_j}\left[(\tilde{E} + \bar{P})\tilde{u}_j - \dfrac{1}{M_A^2}\bar{B}_i\bar{B}_j\tilde{u}_i\right] + \dfrac{\partial}{\partial x_j}\left[\dfrac{\tilde{q}_j}{\mathrm{Pr}\ \mathrm{Re}\ M_s^2(¥-1)} - \dfrac{1}{\mathrm{Re}}\tilde{±}_{ij}\tilde{u}_i\right] - \\[2ex]
\dfrac{\partial}{\partial x_j}\left[\dfrac{¦}{\mathrm{Re}_m M_a^2}\bar{B}_i\left(\dfrac{\partial \bar{B}_i}{\partial x_j} - \dfrac{\partial \bar{B}_j}{\partial x_i}\right)\right] = -\dfrac{\partial}{\partial x_j}\left(\dfrac{1}{¥M_s^2}Q_j + \dfrac{1}{2}J_j + \dfrac{1}{2M_a^2}V_j - \dfrac{1}{M_s^2}G_j\right)
\end{cases}
$$

Equation of state: $\bar{p} = \dfrac{\tilde{T}\bar{\circ}}{¥M_s^2}$

$$E = {}^\circ e + \frac{1}{2}{}^\circ u_i u_i + \frac{1}{2M_a^2}B_i B_i$$

total energy

$$e = \frac{T\rho}{\gamma(\gamma-1)M_s^2}$$

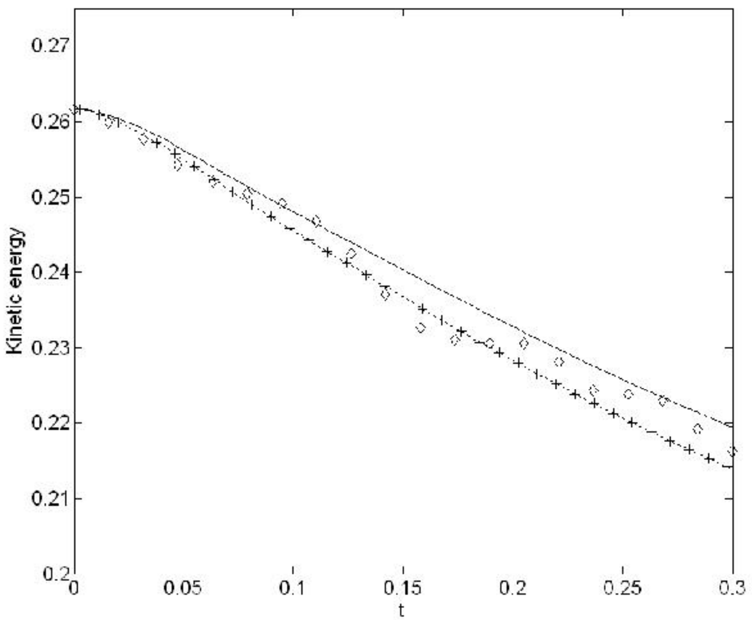
internal energy

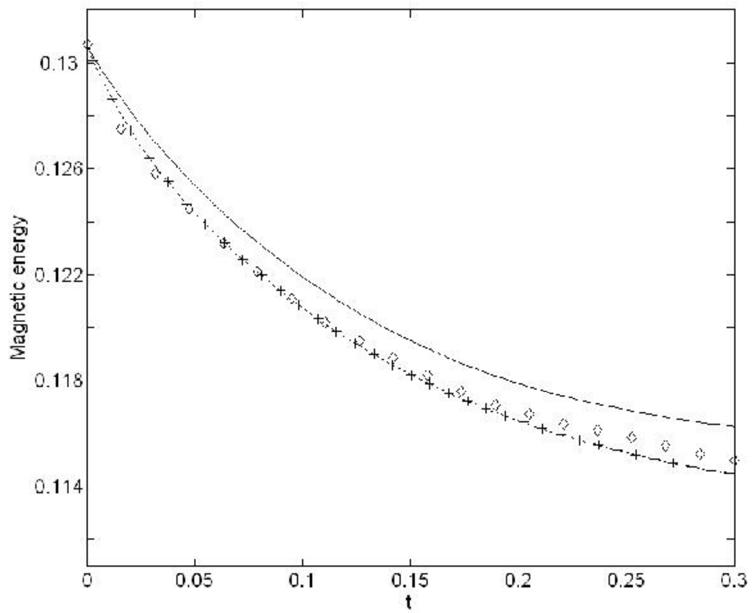
CASE STUDIES

Since compressibility effects and temporal dynamics of temperature defined from the total energy equation depend nontrivially on the Mach number, in this work we consider three cases:
the Mach number $Ms = 0.38$, that is, the flow is moderately compressible;
the Mach number $Ms = 0.70$, when compressibility plays an important role in turbulent fluid flow;
the Mach number $Ms = 1.11$ corresponding to appearance of strong discontinuity in essentially compressible flow.

In all three numerical experiments, the following dimensionless parameters for computations are used: the hydrodynamic Reynolds number $Re = 281$, the microscale (Taylor) Reynolds number $Re\_l = 43$, the magnetic Reynolds number $Re\_m = 10$, the magnetic Mach number $Ma = 1.2$, the Prandtl number $Pr = 1.0$ and the ratio of the specific heats 1.5.

M=0.38

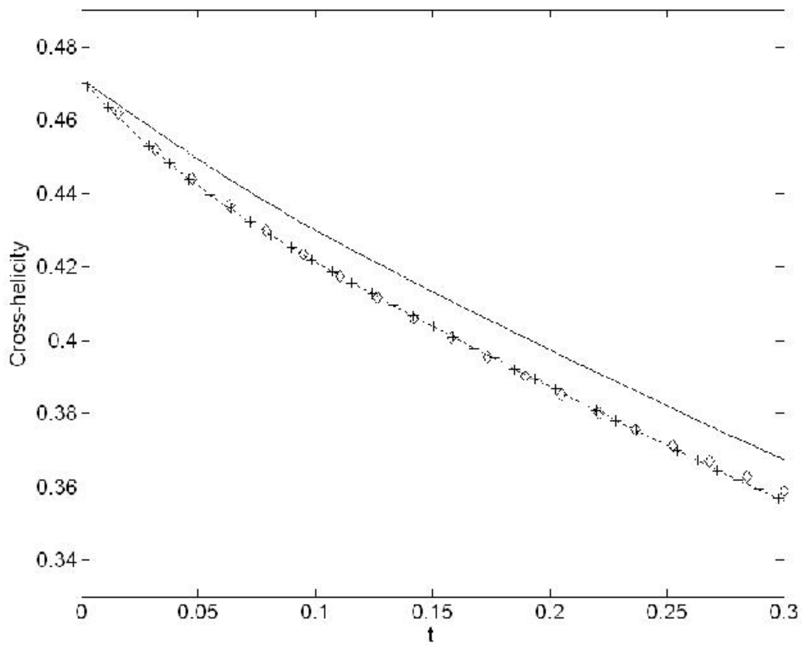


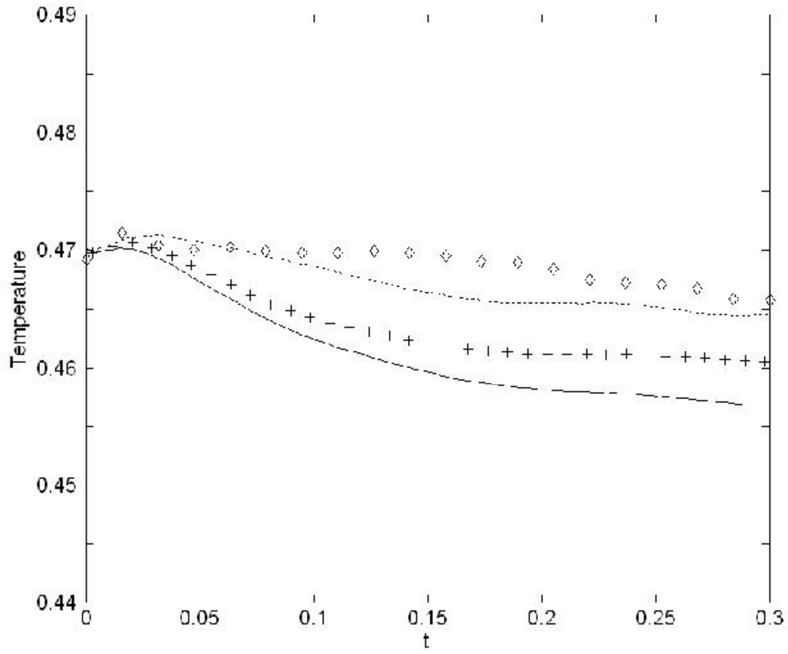


Time dynamics of kinetic and magnetic energy

Case	Curve
DNS	Diamond line
LES without any SGS models	Solid line
LES	Dotted line
LES without energy SGS terms	Marker +

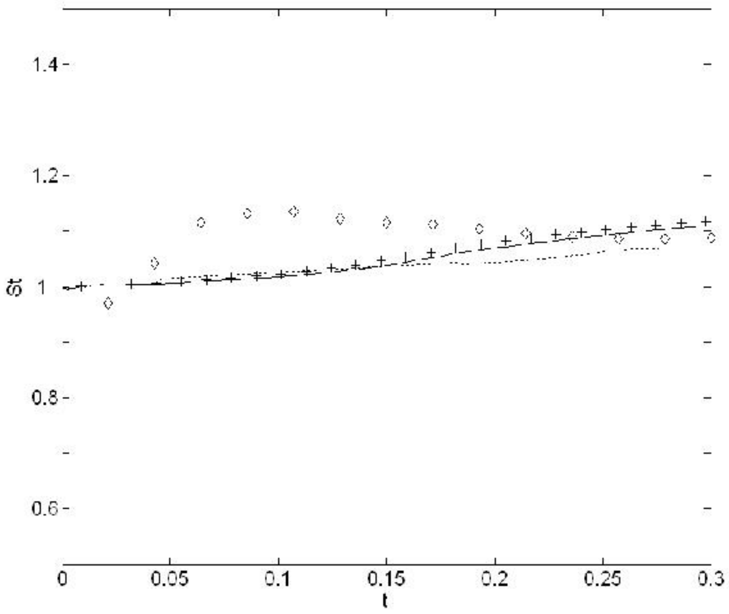
M=0.38

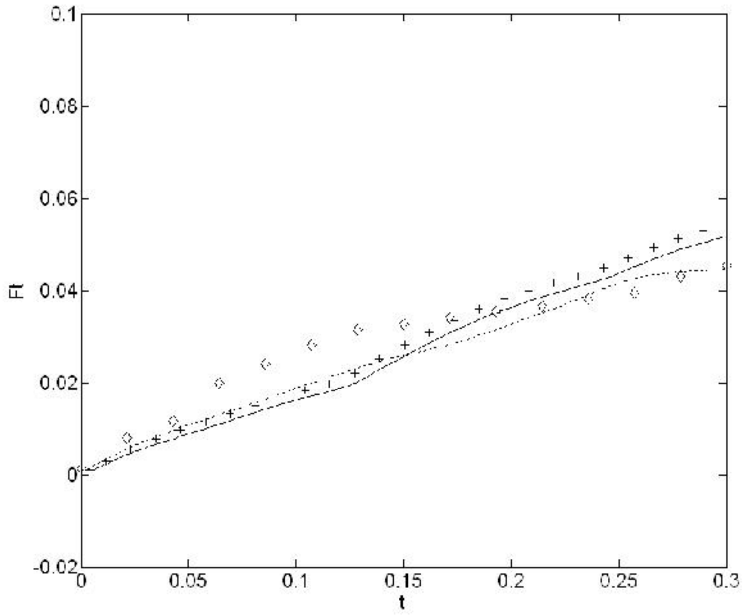




Time evolution of cross-helicity and temperature

M=0.38



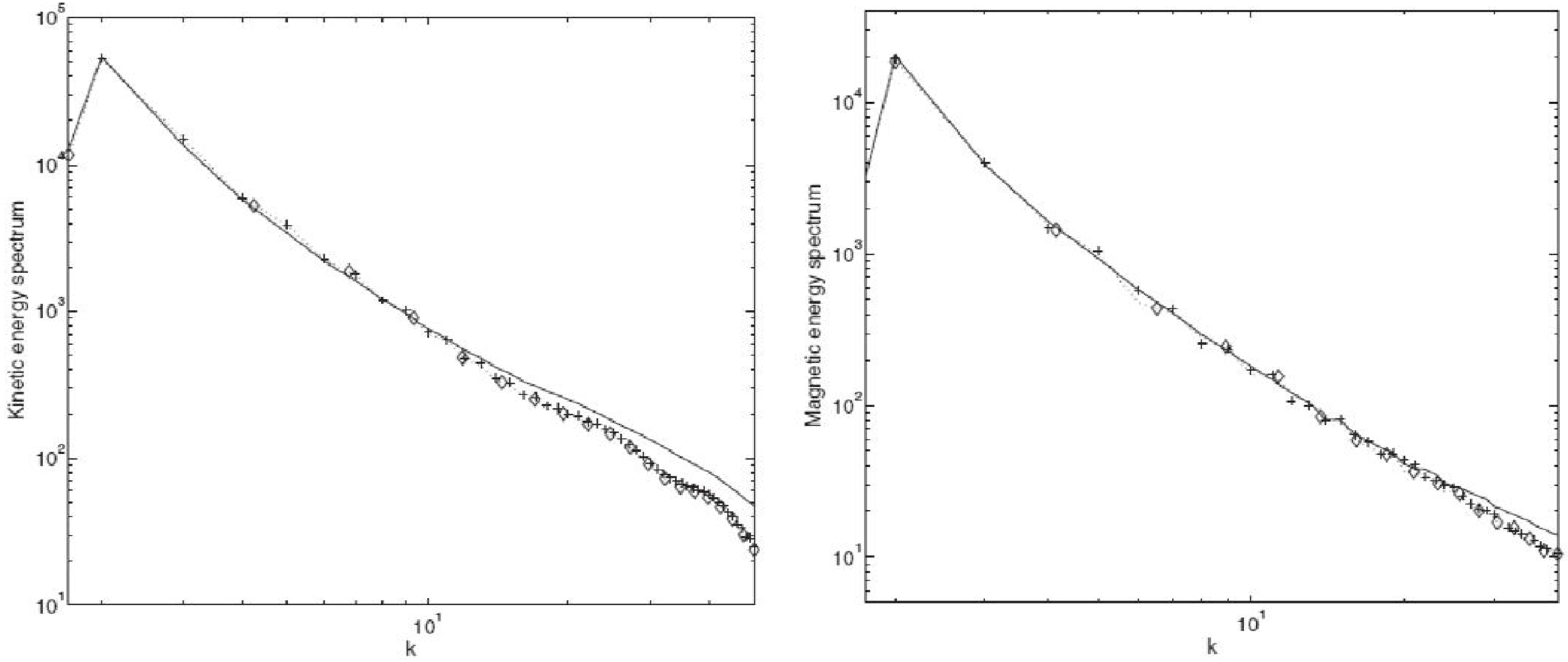


Time dynamics the skewness of the temperature and the parameter Ft

$$St = \frac{<T^3>}{(<T^2>)^{3/2}};$$ - skewness of the temperature

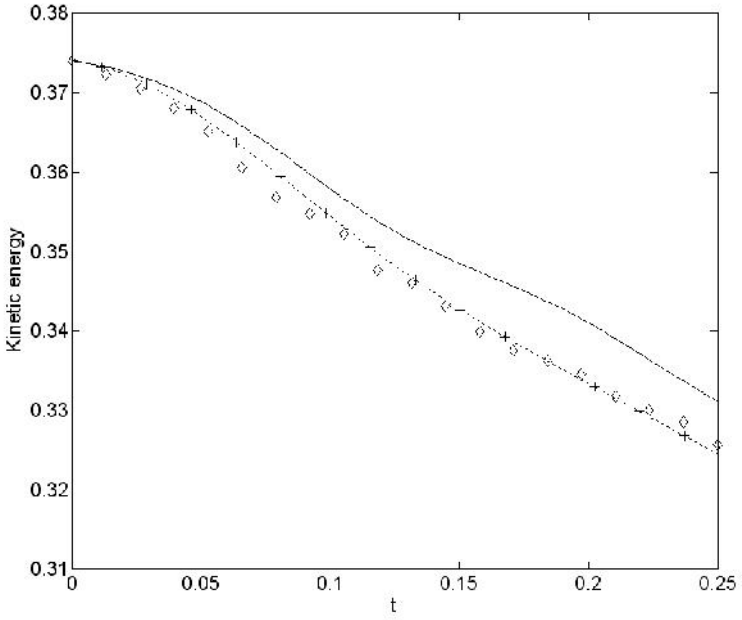
$$Ft = (<(T-<T>)^2>)^{1/2}$$ - parameter, describing temperature fluctuations

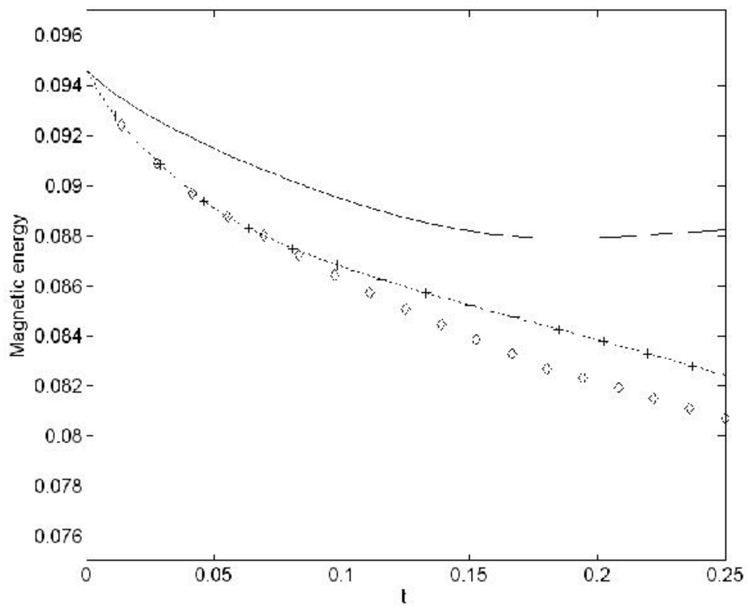
M=0.38



Kinetic and magnetic energy spectra.

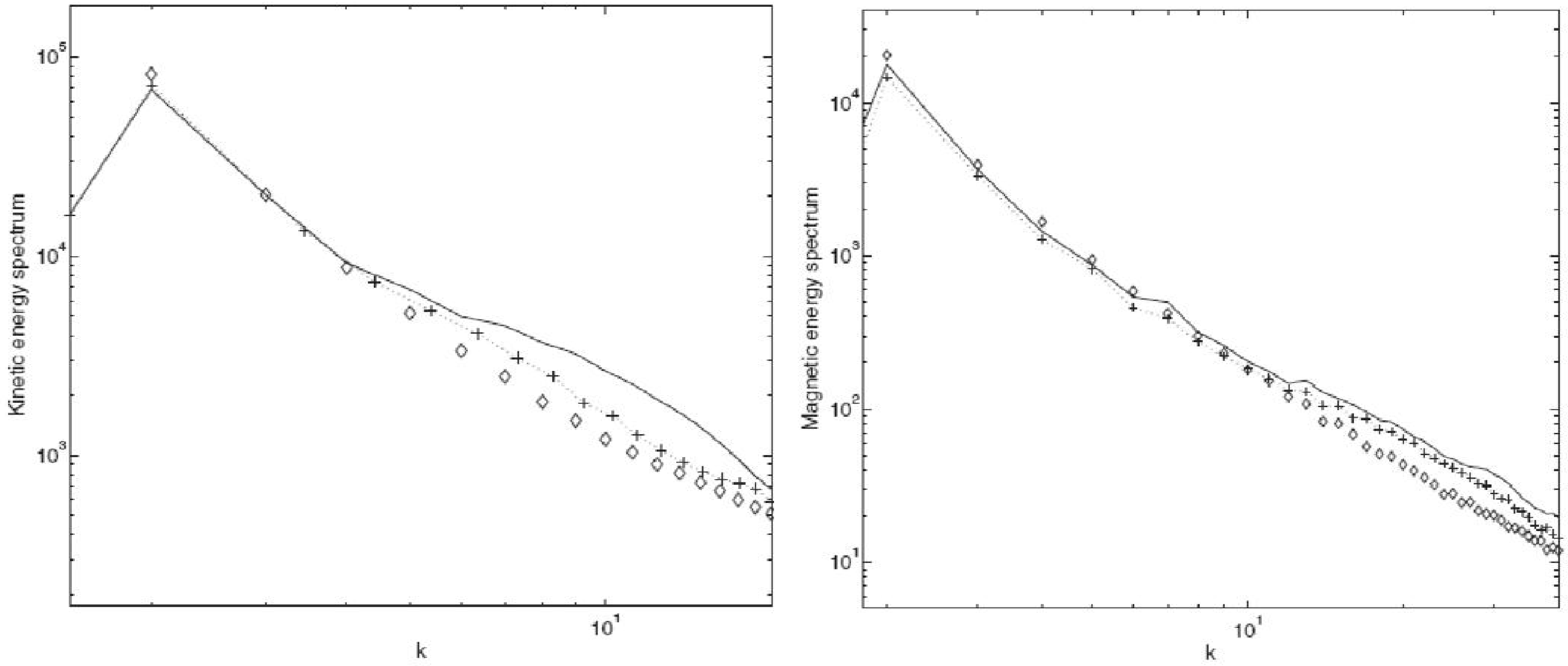
M=0.70





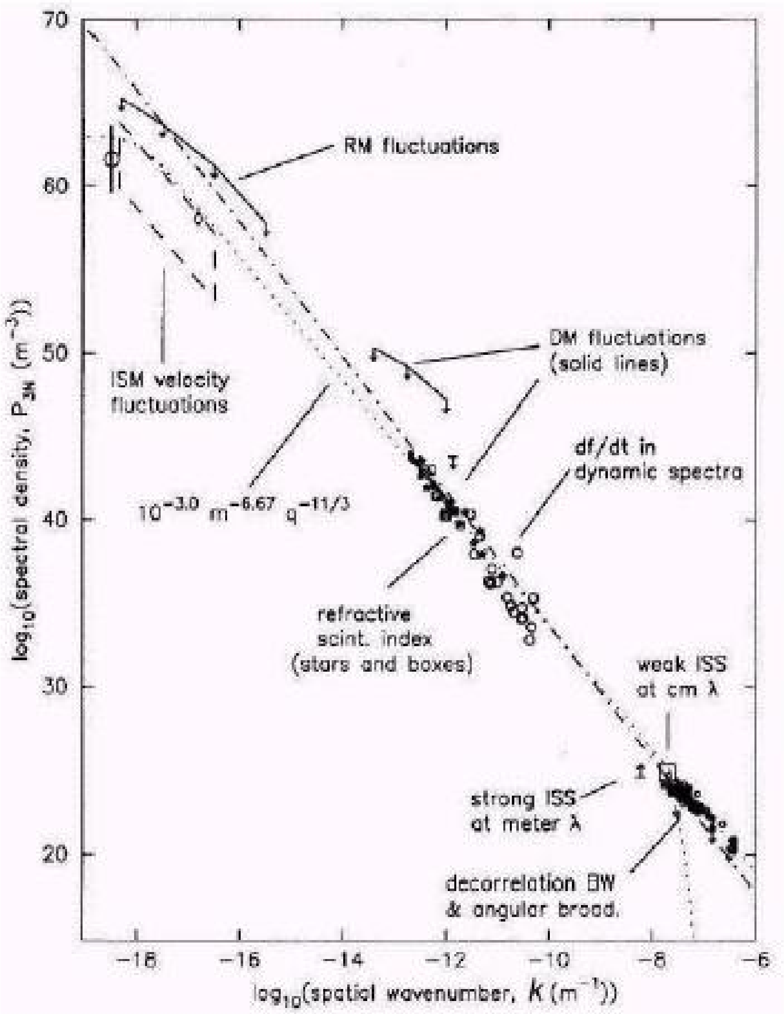
Time dynamics of kinetic and magnetic energy

M=0.70



Kinetic and magnetic energy spectra.

Local Interstellar Medium



Armstrong et al., ApJ (1995), 443:209-221

There is growing interest in observations and explanation of the spectrum of the density fluctuations in the interstellar medium. These fluctuations are responsible for radio wave scattering in the interstellar medium and cause interstellar scintillation fluctuations in the amplitude and phase of radio waves. Kolmogorov-like $k^{-5/3}$ spectrum of density fluctuations have been observed in wide range of scales in the local interstellar medium (from an outer scale of a few parsecs to scales of about 200 km).

Parameters of numerical study of local interstellar medium

For study of compressible MHD turbulence in interstellar, medium we use large eddy simulation (LES) method. Smagorinsky model for compressible MHD case for subgrid-scale parameterization is applied. The Smagorinsky model for compressible MHD turbulence showed accurate results under various range of similarity numbers.

Initial parameters:

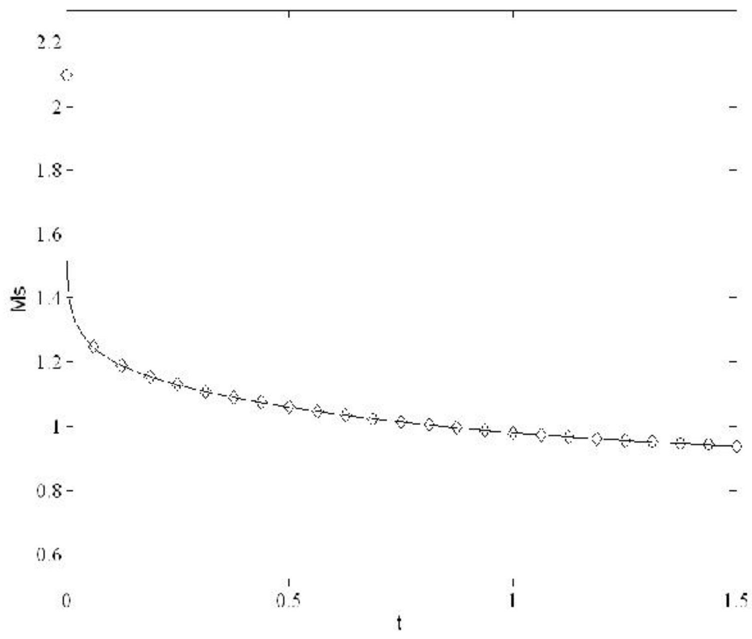
$Re \approx 2000$

$M_s \approx M_A \approx 2.2$

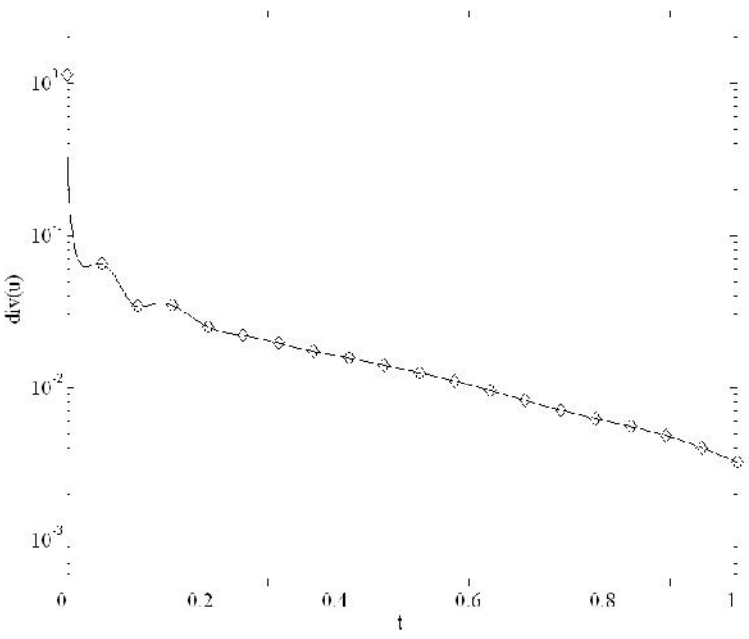
$Re_m \approx 200$ (ambipolar diffusion)

The initial isotropic turbulent spectrum was chosen for kinetic and magnetic energies in Fourier space to be close to k^{-2} with random amplitudes and phases in all three directions. The choice of such spectrum as initial conditions is due to velocity perturbations with an initial power spectrum in Fourier space similar to that of developed turbulence.

Compressibility properties

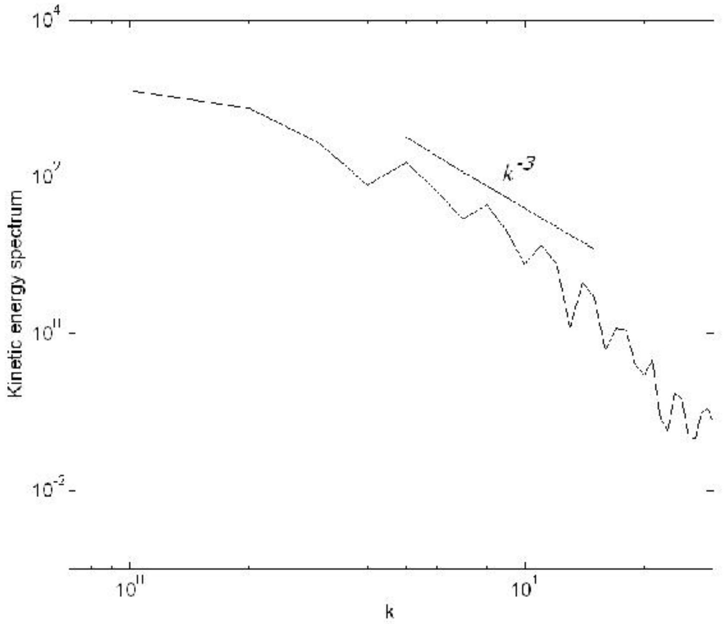


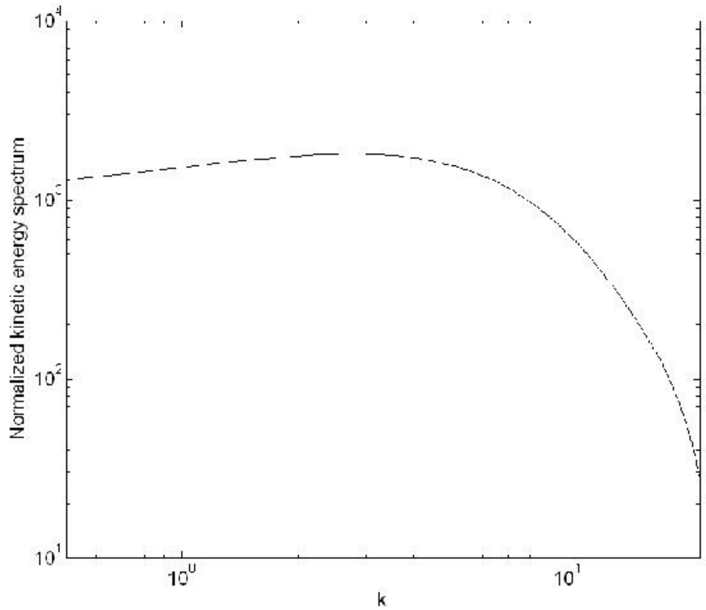
Decay of turbulent small-scale Mach number with time. A transition from a supersonic to a subsonic regime can be observed.



Time dynamics of the velocity divergence. The velocity divergence describing medium compressibility attenuates and the flow becomes weakly compressible with time.

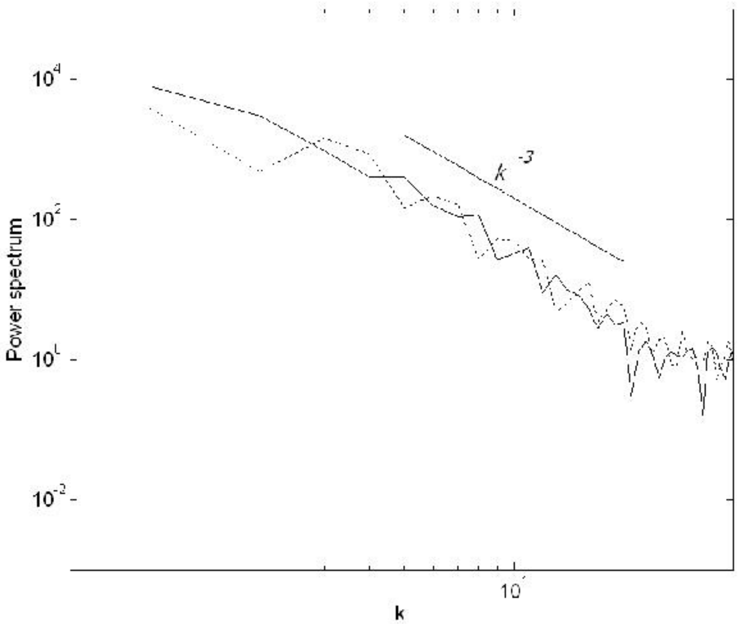
Turbulent spectra in the local interstellar medium

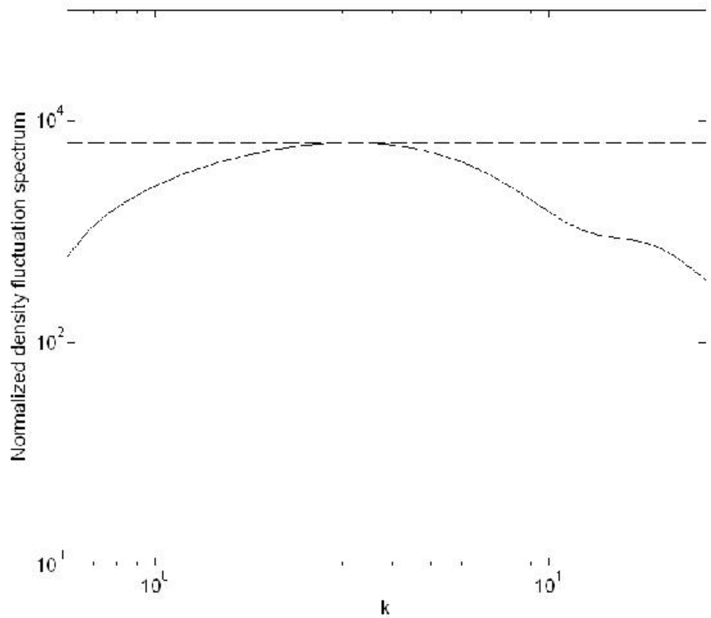




*The kinetic energy spectrum (**left**). Normalized and smoothed spectrum of kinetic energy, multiplied by* $k^{5/3}$ *(**right**). Notice that the spectrum is close to* k^{-3} *in a forward cascade regime of decaying turbulence. However, there is well-defined inertial Kolmogorov-like range of* $k^{-5/3}$

Turbulent spectra in the local interstellar medium





The density spectrum is the solid line and the density fluctuations spectrum is the dot line (**left**). Normalized and smoothed spectrum of density fluctuations, multiplied by $k^{5/3}$ (**right**). Both graphs (in the left figure) have spectral index close to k^{-3}. Moreover, there is well-defined inertial Kolmogorov-like range of $k^{-5/3}$ that confirms observation data.

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