Using Taylor-Green symmetries in 3D Magnetohydrodynamic

Marc Brachet

Turbulence workshop, « Energy Cascade and Dissipation in Astrophysical turbulent Plasmas » Meudon, CIAS, 26-29 May 2015

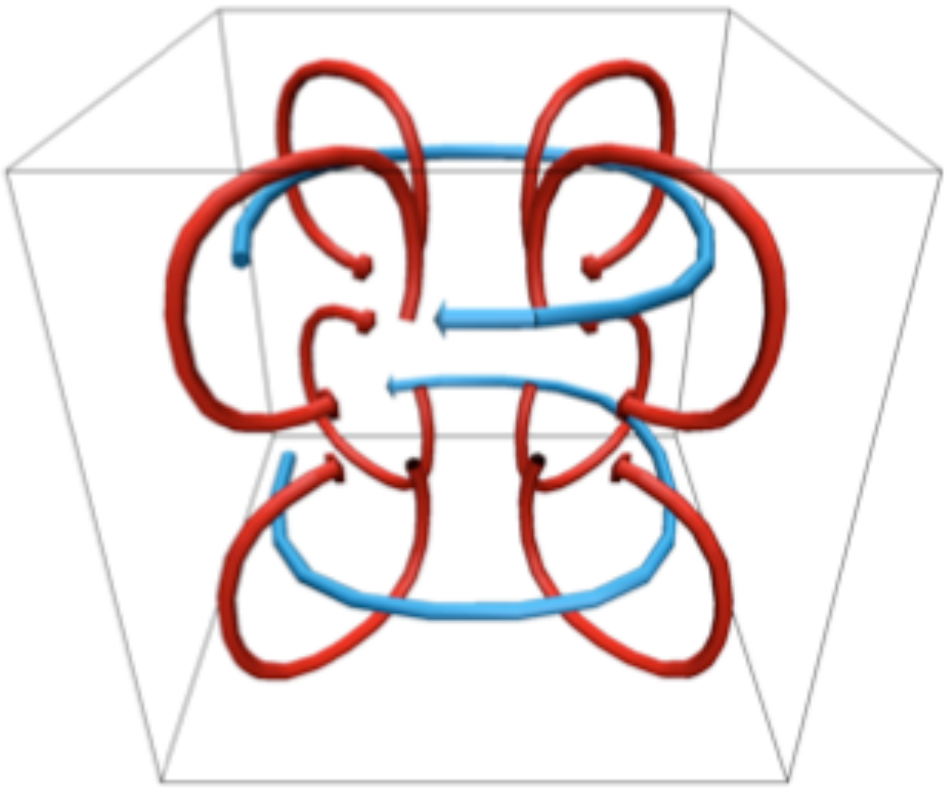
Meudon 26/05/2015

Plan of Talk

- Von-Karman v.s. Taylor-Green?
- Dynamo: keep the mirrors symmetries (that confine the flow) and drop the rotations (that do not allow magnetic dipoles)
- Singularities: use all the symmetries
- Symmetries and MHD turbulence
- Discussion

TYG vs. VKS

Similar qualitative aspect: counter rotation in both cases, but:



Cubic (impermeable) box

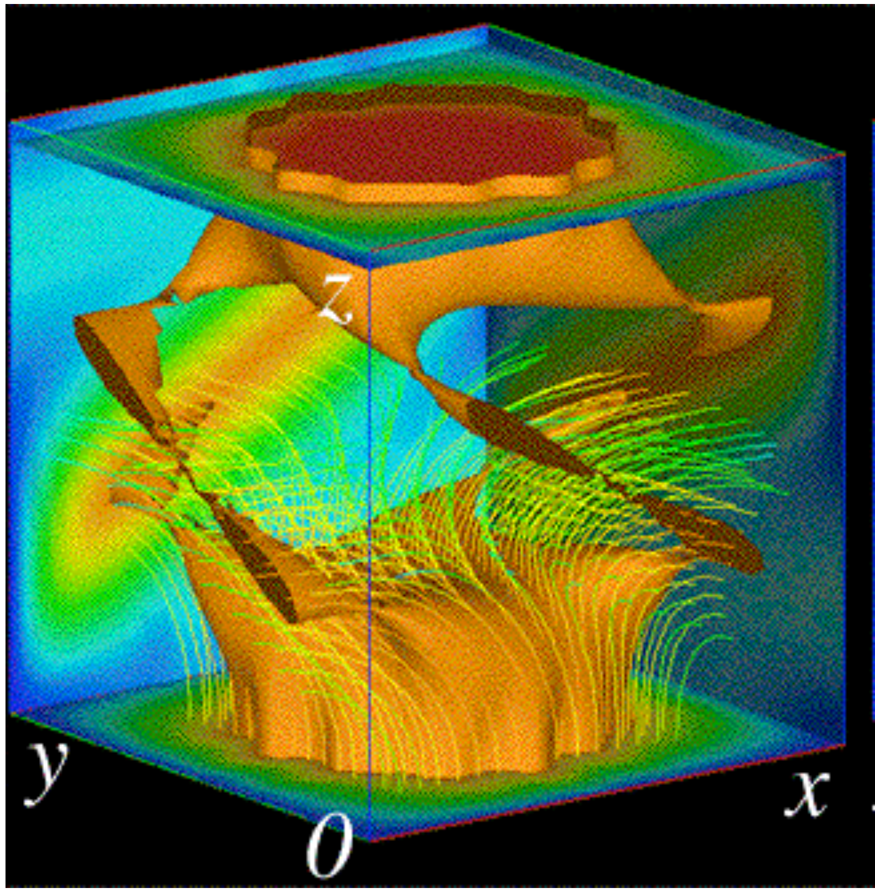
free-slip boundaries

cylindrical box

no-slip boundaries

What are the TYG Symmetries?

- Flow is 2-Pi periodic
- Impermeable box: x=0,Pi; y=0,Pi and z=0,Pi are planes of mirror symmetry
- Rotation by Pi around axis x=z=Pi/2 and y=z=Pi/2
- Rotation by Pi/2 around the axis x=y=Pi/2



Dynamo

following results are from:

Giorgio Krstulovic, Gentien Thorner, Julien-Piera Vest,
Stephan Fauve, and Marc Brachet,
Axial dipolar dynamo action in the Taylor-Green vortex ,
Phys. Rev. E 84, 066318 (2011)

Symmetries are bad for dynamo!

Dynamo action in the Taylor–Green vortex near threshold

C. Nore[a)] and M. E. Brachet
Laboratoire de Physique Statistique, CNRS URA 1306, ENS Ulm, 24 Rue Lhomond, 75231 Paris Cedex 05, France

H. Politano and A. Pouquet
Observatoire de la Côte d'Azur, CNRS URA 1362, BP 229, 06304 Nice-Cedex 4, France

(Received 17 April 1996; accepted 10 October 1996)

Dynamo action is demonstrated numerically in the forced Taylor–Green (TG) vortex made up of a confined swirling flow composed of a shear layer between two counter-rotating eddies, corresponding to a standard experimental setup in the study of turbulence. The critical magnetic Reynolds number above which the dynamo sets in depends crucially on the fundamental symmetries of the TG vortex. These symmetries can be broken by introducing a scale separation in the flow, or by letting develop a small non-symmetric perturbation which can be either kinetic and magnetic, or only magnetic. The nature of the boundary conditions for the magnetic field (either conducting or insulating) is essential in selecting the fastest growing mode; implications of these results to a planned laboratory experiment are briefly discussed.
[S1070-664X(97)02501-9]

Periodicity is good!

Dynamo action at low magnetic Prandtl numbers: mean flow versus fully turbulent motions

Y Ponty[1], **P D Mininni**[2], **J-F Pinton**[3], **H Politano**[1] **and A Pouquet**[2]

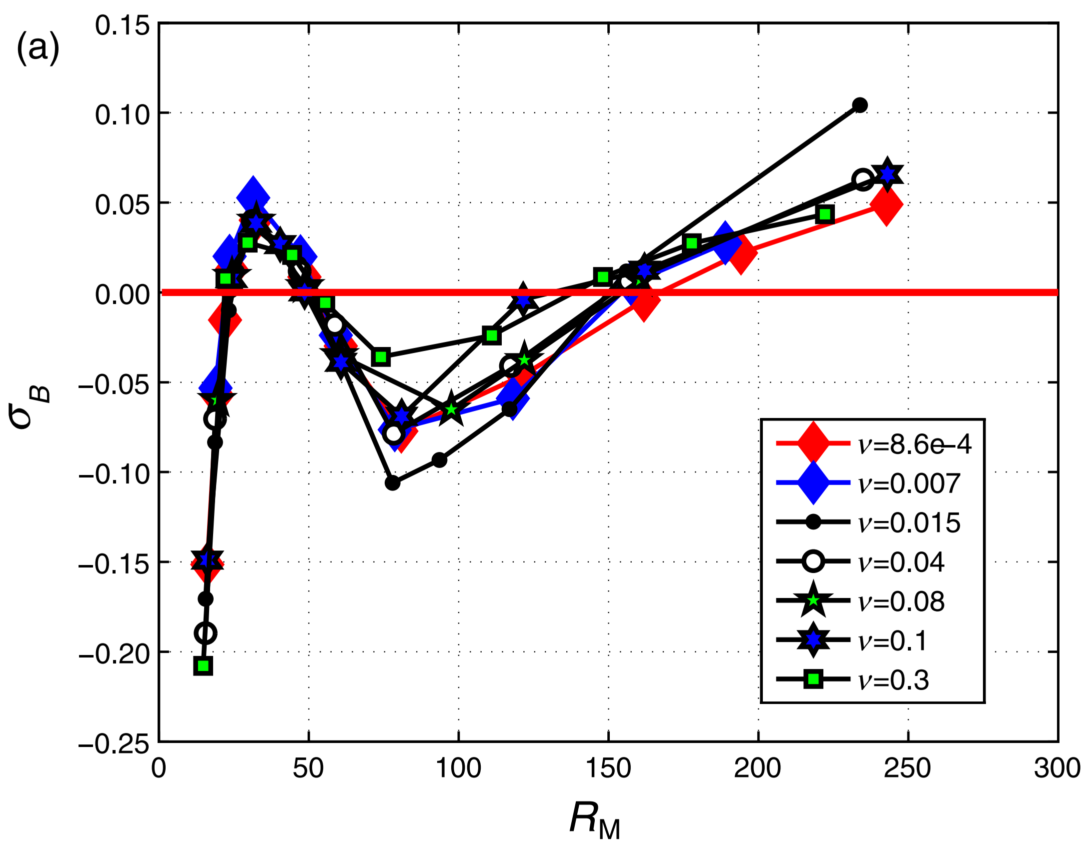
[1] CNRS UMR6202, Laboratoire Cassiopée, Observatoire de la Côte d'Azur, BP 4229, Nice Cedex 04, France

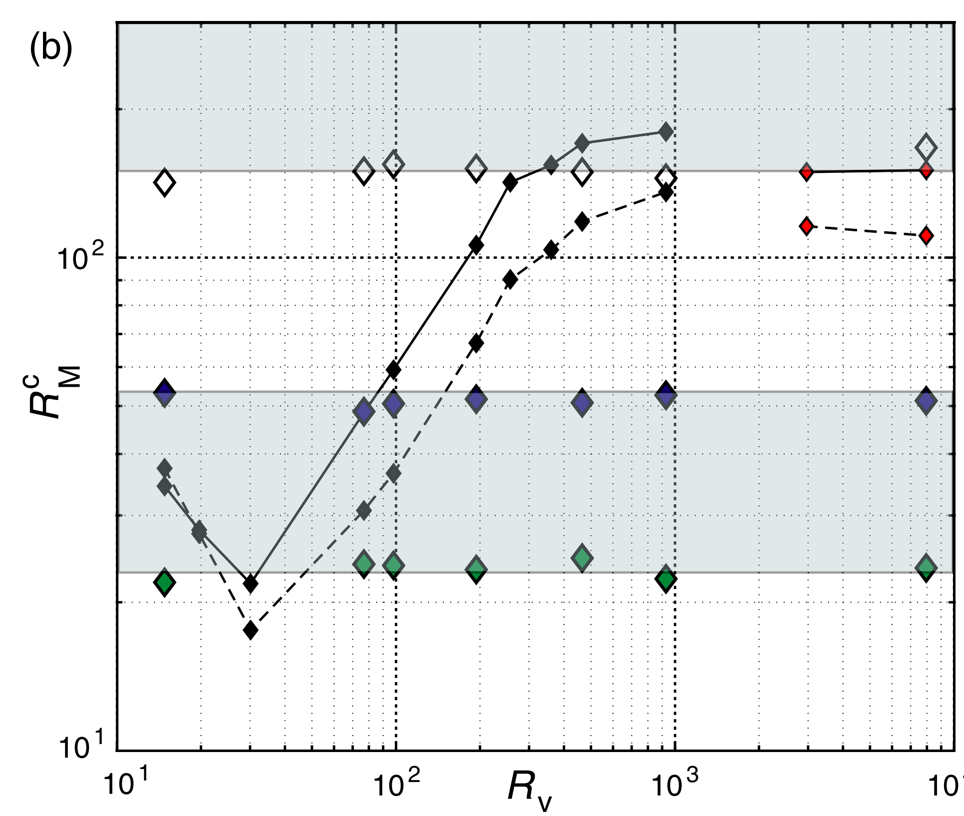
[2] NCAR, P O Box 3000, Boulder Colorado 80307-3000, USA

[3] CNRS UMR5672, Laboratoire de Physique, École Normale Supérieure de Lyon, 46 Allée d'Italie, 69007 Lyon, France

E-mail: Yannick.PONTY@obs-nice.fr

New Journal of Physics **9** (2007) 296
Received 17 January 2007
Published 31 August 2007
Online at http://www.njp.org/
doi:10.1088/1367-2630/9/8/296





A (simple?) idea

- Keep the mirror symmetries that do confine the flow
- Drop the axis of rotation that do not allow for dipolar fields within the impermeable box
- How to generalize for magnetic field?

Mirrors

$$\mathbf{S}^1(r_1, r_2, r_3) = (-r_1, r_2, r_3), \tag{9}$$
$$\mathbf{S}^2(r_1, r_2, r_3) = (r_1, -r_2, r_3), \tag{10}$$
$$\mathbf{S}^3(r_1, r_2, r_3) = (r_1, r_2, -r_3). \tag{11}$$

Note that $\mathbf{S}^\alpha$ is its own inverse.

The action of the reflection operation on a vector field $\mathbf{h}(\mathbf{r})$ is defined by

$$\mathbf{R}^\alpha(\mathbf{h}(\mathbf{r})) = \mathbf{S}^\alpha \mathbf{h}(\mathbf{S}^\alpha \mathbf{r}) \tag{12}$$

which explicitly reads, in the case of the the $z = 0$ plane:

$$\mathbf{R}^z \begin{pmatrix} h_x(x,y,z) \\ h_y(x,y,z) \\ h_z(x,y,z) \end{pmatrix} = \begin{pmatrix} h_x(x,y,-z) \\ h_y(x,y,-z) \\ -h_z(x,y,-z) \end{pmatrix}. \tag{13}$$

The action of the reflection operation on a vector field $\mathbf{h}(\mathbf{r})$ is defined by

$$\mathbf{R}^\alpha(\mathbf{h}(\mathbf{r})) = \mathbf{S}^\alpha \mathbf{h}(\mathbf{S}^\alpha \mathbf{r}) \tag{12}$$

Mirrors

$$
\begin{aligned}
\partial_t \bar{\mathbf{v}} + (\nabla \times \bar{\mathbf{v}}) \times \bar{\mathbf{v}} &= -\nabla(\bar{P} + \frac{1}{2}\bar{\mathbf{v}}^2) + (\nabla \bar{\mathbf{b}}) \times \bar{\mathbf{b}} \\
&\quad + \nu \Delta \bar{\mathbf{v}} + \mathbf{R}^\alpha \mathbf{f} \qquad (17) \\
\partial_t \bar{\mathbf{b}} &= \nabla \times (\bar{\mathbf{v}} \times \bar{\mathbf{b}}) + \eta \Delta \bar{\mathbf{b}}, \qquad (18)
\end{aligned}
$$

with $\bar{P}(\mathbf{r}) = P(\mathbf{S}^\alpha \mathbf{r})$. Therefore the only symmetry of the velocity that is compatible with the MHD equations (1-2) is $\mathbf{R}^\alpha \mathbf{f} = \mathbf{f}$ and $\bar{\mathbf{v}} = \mathbf{v}$. This can be easily understood with the following simple geometrical argument.

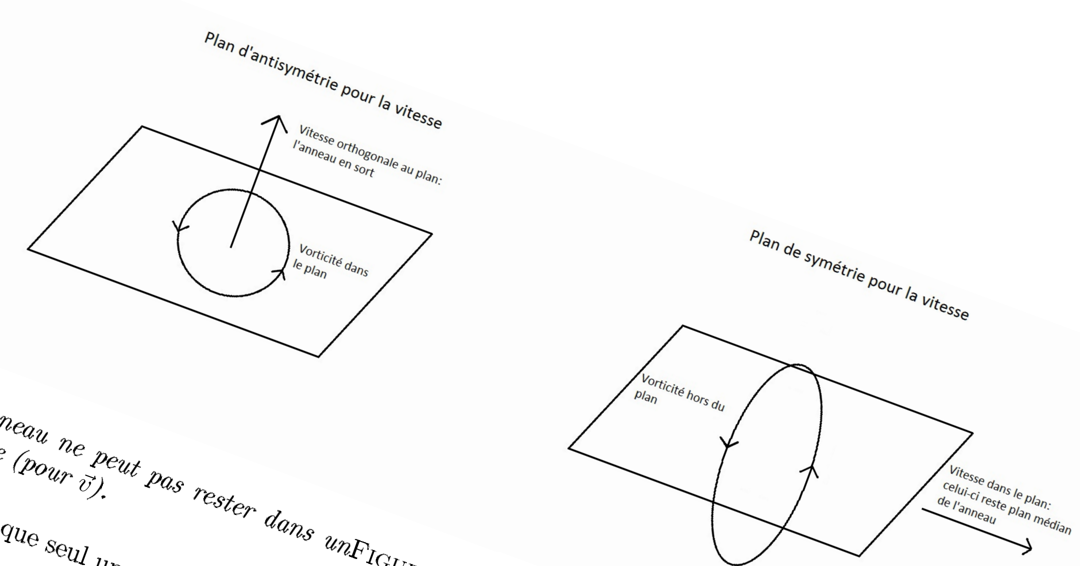


FIGURE 3 – *L'anneau ne peut pas rester dans un plan d'antisymétrie (pour $\vec{v}$).*

FIGURE 4 – *Un plan de symétrie (pour $\vec{v}$) de l'anneau le reste à tout instant.*

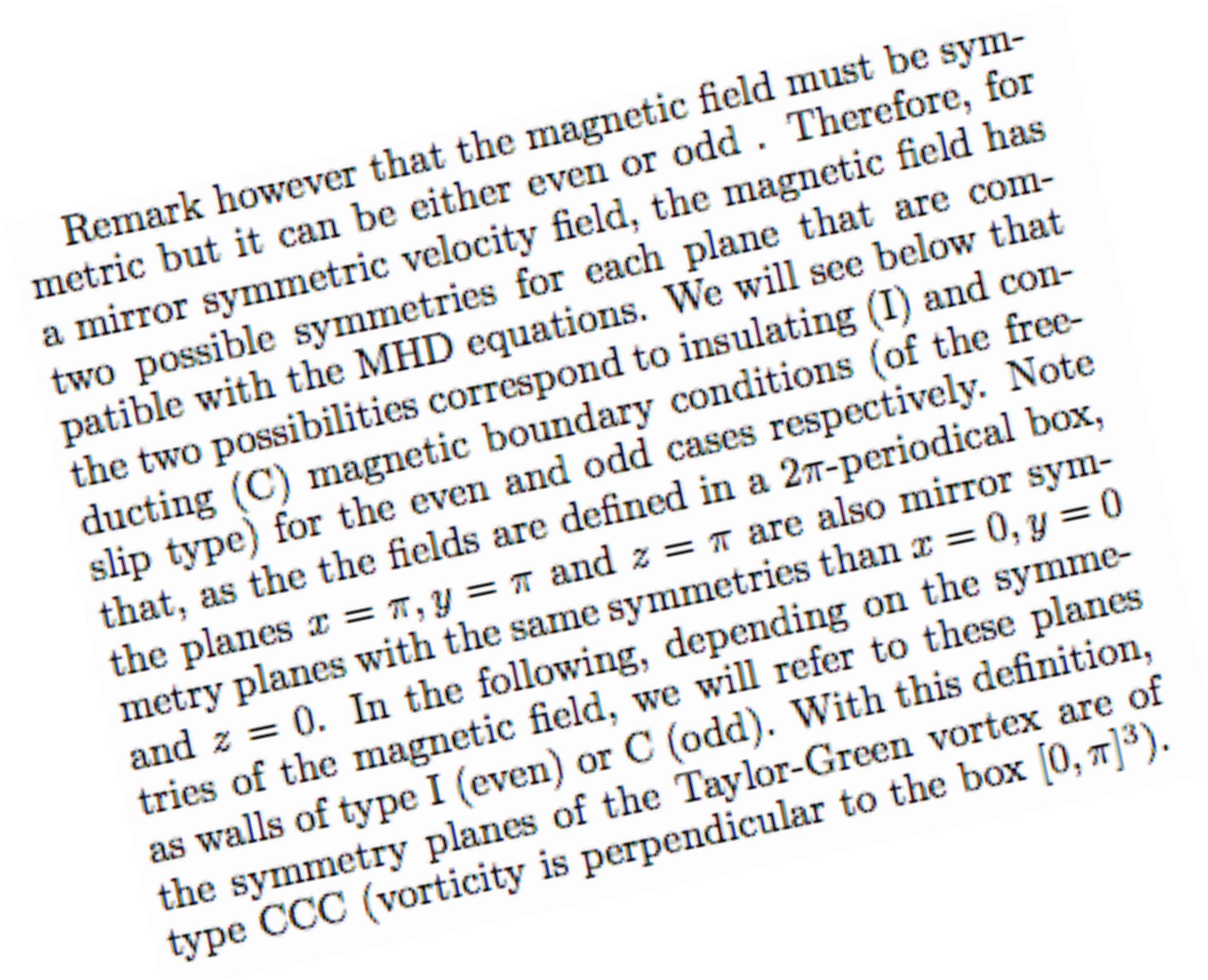
On retrouve bien que seul un plan de symétrie pour $\vec{v}$ peut être conservé au cours du temps.

Mirrors

mag. boundary is either

I or

C



Remark however that the magnetic field must be symmetric but it can be either even or odd . Therefore, for a mirror symmetric velocity field, the magnetic field has two possible symmetries for each plane that are compatible with the MHD equations. We will see below that the two possibilities correspond to insulating (I) and conducting (C) magnetic boundary conditions (of the free-slip type) for the even and odd cases respectively. Note that, as the the fields are defined in a 2π-periodical box, the planes $x = \pi, y = \pi$ and $z = \pi$ are also mirror symmetry planes with the same symmetries than $x = 0, y = 0$ and $z = 0$. In the following, depending on the symmetries of the magnetic field, we will refer to these planes as walls of type I (even) or C (odd). With this definition, the symmetry planes of the Taylor-Green vortex are of type CCC (vorticity is perpendicular to the box $[0, \pi]^3$).

Projectors

Let us now define the projectors into symmetric functions with respect to the planes $r_i = 0$ and $r_i = \pi$,

$$\mathbf{Q}_s^\alpha = \frac{1}{2}(I - s\mathbf{R}^\alpha), \tag{19}$$

where α stands for x, y, z, $\mathbf{R}^\alpha$ is defined by Eq.(12) and $s = \pm 1$. Note that by construction $\mathbf{R}^\alpha(\mathbf{Q}_s^\alpha \mathbf{h}) = -s\mathbf{Q}_s^\alpha \mathbf{h}$ and therefore $Q_{x_i}^s \mathbf{h}$ is an even vector (if $s = +1$) or odd vector ($s = -1$) with respect to the planes $r_\alpha = 0$ and $r_\alpha = \pi$.

$$\mathbf{P}^{\vec{s}} = \mathbf{Q}_{s_1}^1 \mathbf{Q}_{s_2}^2 \mathbf{Q}_{s_3}^3, \tag{20}$$

with $\vec{s} = (s_x, s_y, s_z) \in \{-1, 1\}^3$. Note that by construction

$$\mathbf{P}^{(-1,-1,-1)}\mathbf{v}^{\mathrm{TG}} = \mathbf{v}^{\mathrm{TG}}. \tag{21}$$

ditions. If the magnetic field is even with respect to one of the mirror-symmetry planes, we call this plane a *insulating* (I) wall because the current $\mathbf{j} = \nabla \times \mathbf{b}$ is parallel to (or contained in) the wall [17]. Analogously if $\mathbf{b}$ is odd the wall is called *conducting* (the current is perpendicular to (or crosses) the wall.

I or

C

Velocity must be C magnetic boundaries: I/C

- Flow is confined
- Isolating/Conducting boundaries perpendicular to x, y or z
- 8 possible cases: 2x2x2, but 2 couples among them are exchanged by Pi/2 rotation around the x=y=Pi/2 axis
- Therefore there are 6 independent cases!

Numerical procedure

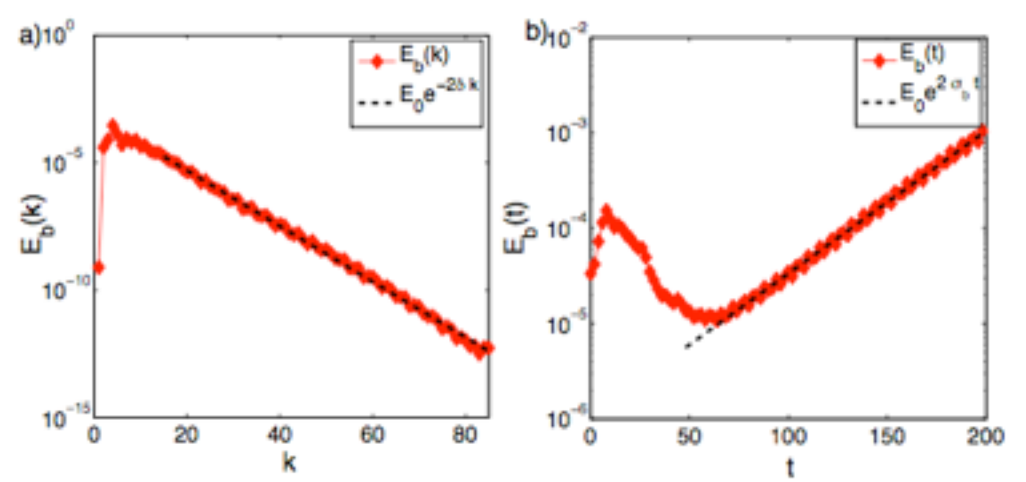


FIG. 1: a) Magnetic energy spectrum of a dynamo run: case III with $\mathrm{Re} = 30$, $\mathrm{Re_m} = 80$. b) Corresponding temporal evolution of magnetic energy (3). The fits used to determine $\sigma_b = 0.017$ and $\delta k_{\max} = 10.72$ (for the present simulation) are displayed as (dashed) straight lines of Figs. a) and b).

GHOST [19], that is dealiased by spherical spectral truncation using the 2/3-rule [20]. Thus a run with resolution N^3 has a maximum wavenumber $k_{\max} = N/3$. Resolutions used in this works vary from 64^3 to 256^3. The equations are evolved in time using a second-order Runge-Kutta method, and the code is fully parallelized with the message passing interface MPI library. We implemented into GHOST both the constant velocity forcing (7) and the projectors (20).

The TG vortex (8) is used as initial data for Eq.(1), eventually adding a small non-symmetric random part when studying symmetry breaking (see below section III B). A small, spectrally band-limited random seed of given magnetic energy is used as initial data for Eq.(2).

Spontaneous confinment breaking

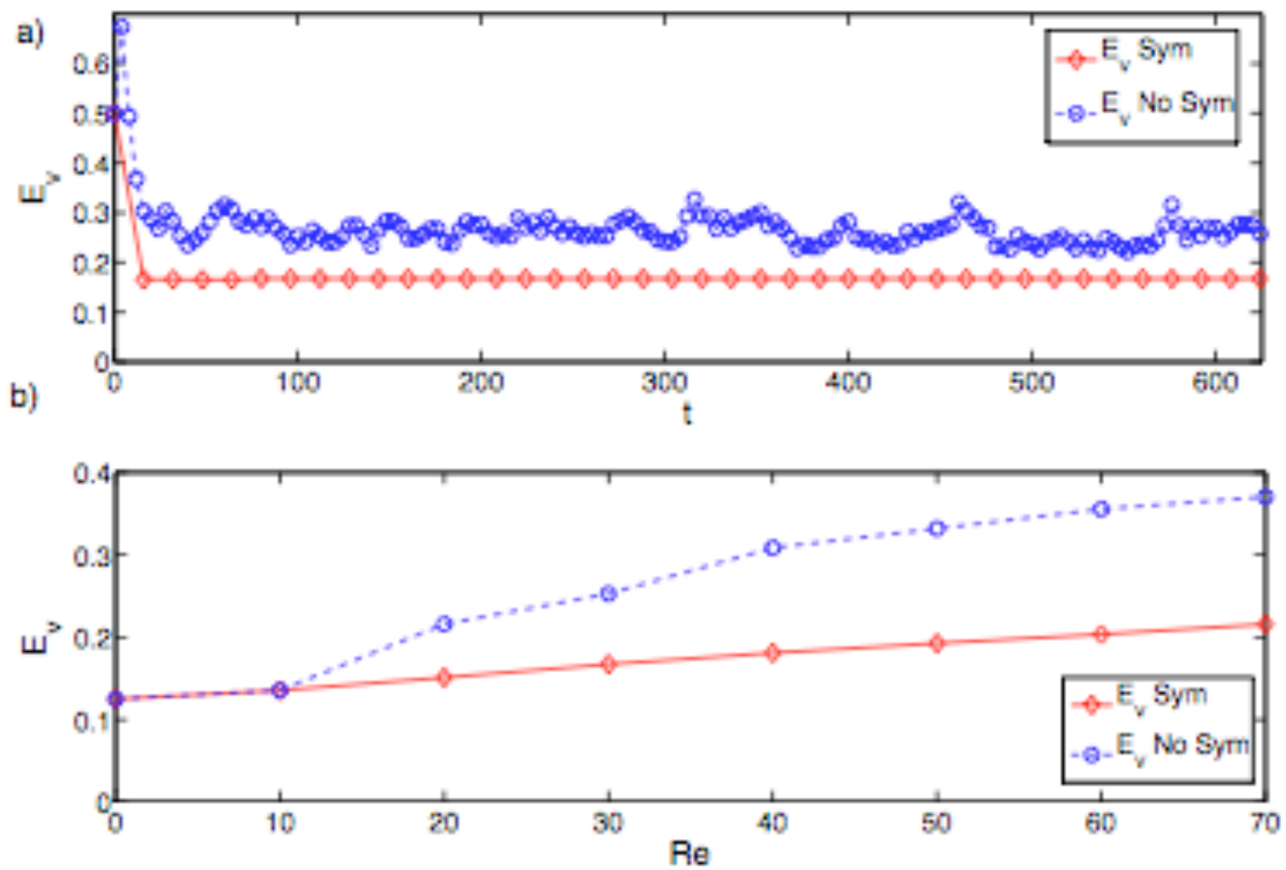


FIG. 2: (Color online) a) Temporal evolution of kinetic energy with and without confinement (21) imposed at Re = 30. b) Dependence of kinetic energy on the Reynolds number for the symmetric solid (red) line and non-symmetric dashed (blue) line runs (time-averaged over statistically stationary values).

B=0 run:
with and without
confinement projection

Spontaneous Pi/2 rotation breaking

B=0 run:
with confinement projection

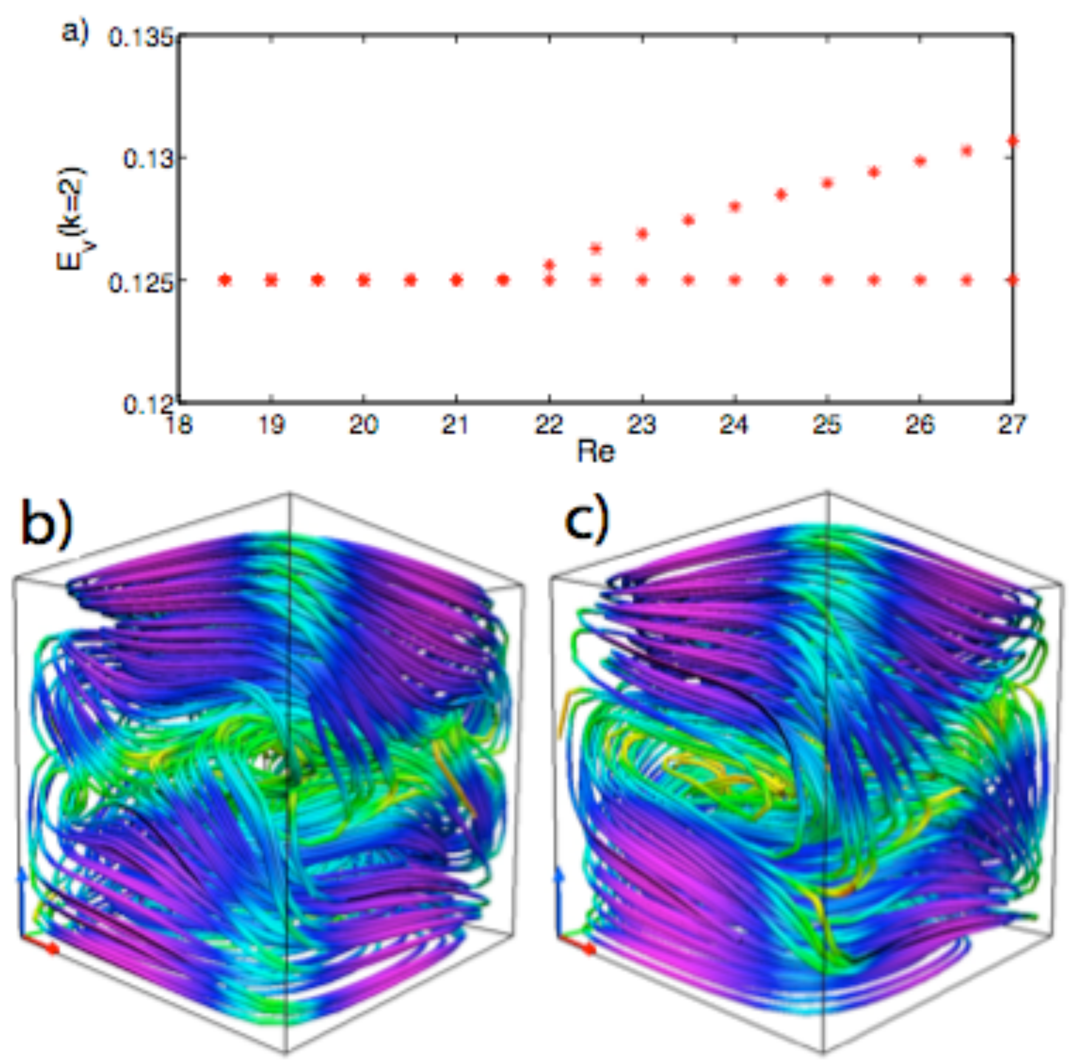


FIG. 3: (Color online) a) Bifurcation diagram: kinetic energy $E_v(k=2)$ as function of Re. A pitchfork bifurcation is clearly present ($E_v(k=2)$ is quadratic in the bifurcating mode $\mathbf{v}^{\mathrm{PF}}$). b-c) Visualization of Taylor-green stationary states at Re = 30: non-bifurcated (b) and bifurcated (c) velocity fields.

$$\mathbf{v}^{\mathrm{PF}} = \begin{pmatrix} \sin(x)\cos(y)\cos(z) \\ \cos(x)\sin(y)\cos(z) \\ -2\cos(x)\cos(y)\sin(z) \end{pmatrix}.$$

Magnetic boundary conditions and dynamo threshold

Case	ICI	ICC	IIC	III	CCC	CCI
Re_m^c	9	26	66	73	231	254

TABLE I: Critical magnetic Reynolds number for the different walls and symmetric and non-symmetric cases at Re = 30 Values obtained by linear fit of σ_b.

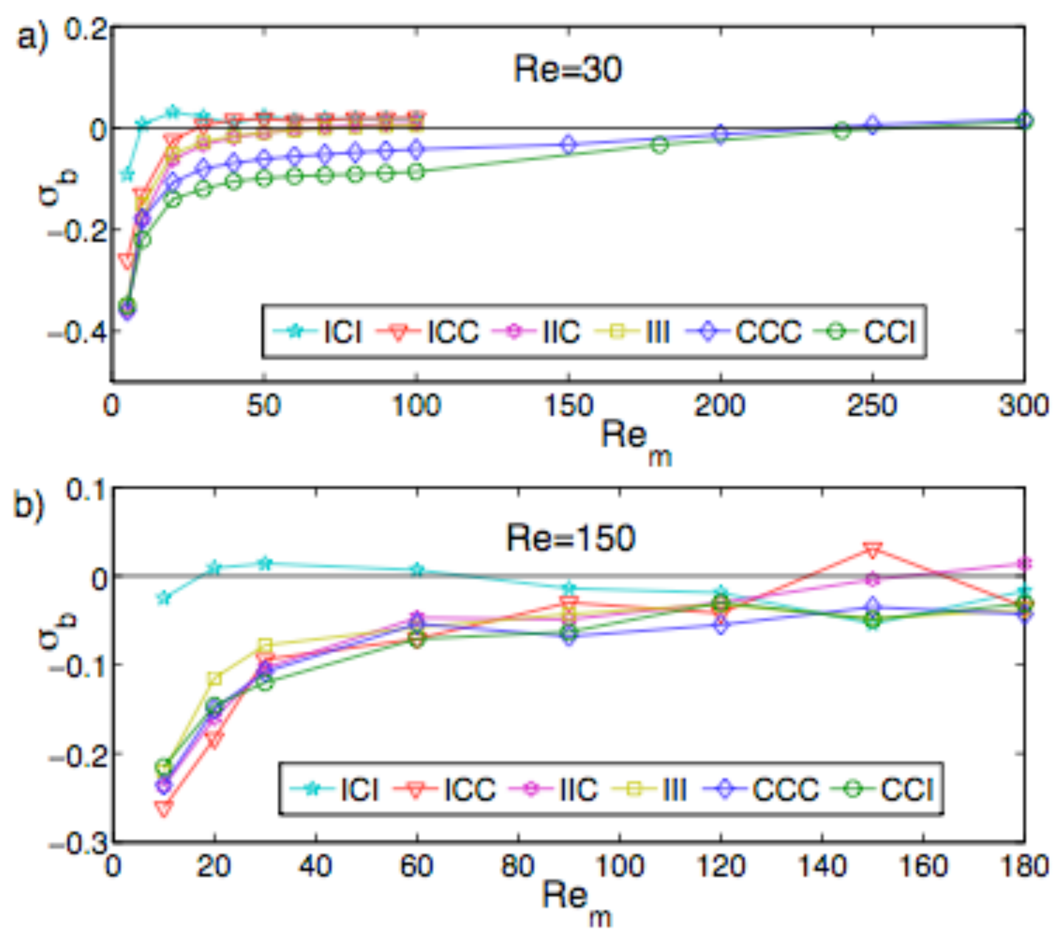


FIG. 5: (Color online) Dynamo growth rates σ_b (symmetric velocity field) as function of the magnetic Reynolds number Re_m for the 6 possible symmetries of the magnetic field at kinetic Reynolds number Re = 30 (a) and Re = 150 (b).

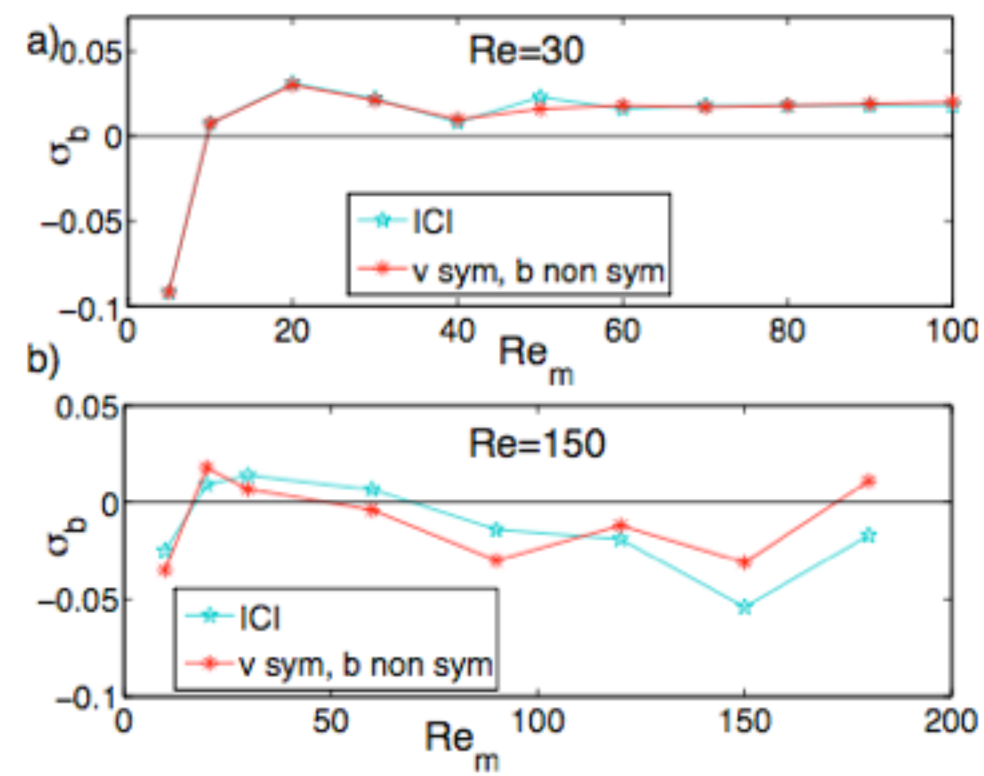


FIG. 6: (Color online) Dynamo growth rates σ_b (symmetric velocity field) obtained with a non-symmetric magnetic field compared with the CIC magnetic symmetric case. Kinetic Reynolds number Re = 30 (a) and Re = 150 (b).

ICI, ICC, IIC, CCC and CCI cases

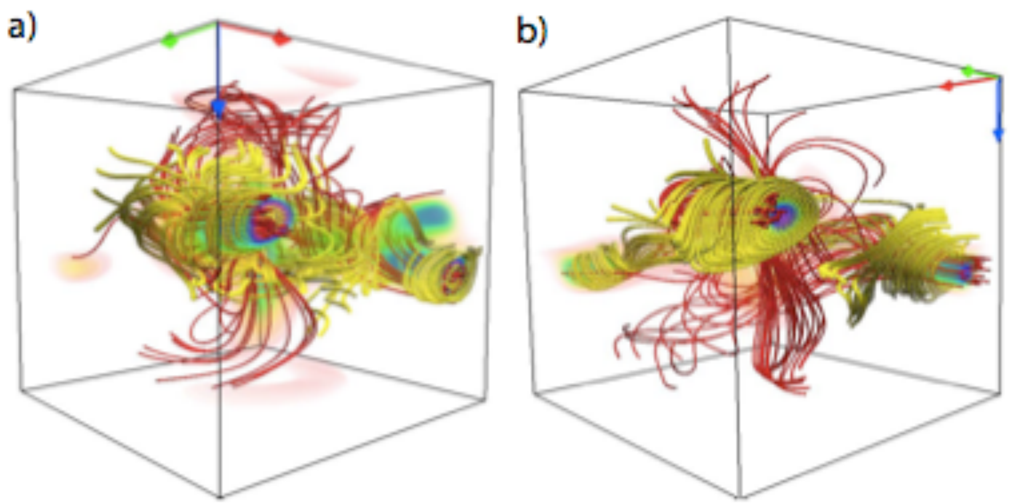


FIG. 7: (Color online) 3D visualizations (magnetic field in red, current in yellow and density plot of highest magnetic energy zones) of the growing modes: (a) Case v symmetric and b non-symmetric, $\mathrm{Re}_m = 10$. (b) ICI case, $\mathrm{Re}_m = 10$.

Case	ICI	ICC	IIC	III	CCC	CCI
Re^c_m	9	26	66	73	231	254

TABLE I: Critical magnetic Reynolds number for the different walls and symmetric and non-symmetric cases at Re = 30. Values obtained by linear fit of σ_b.

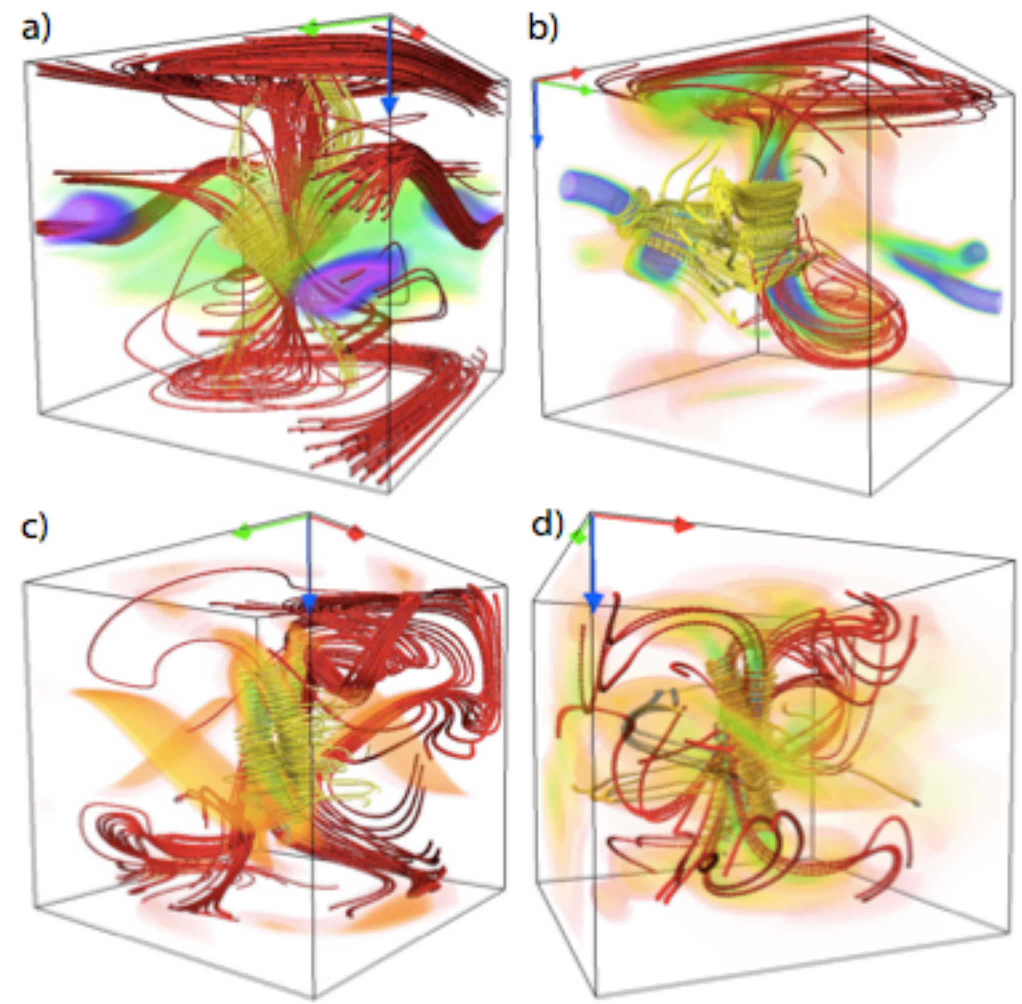
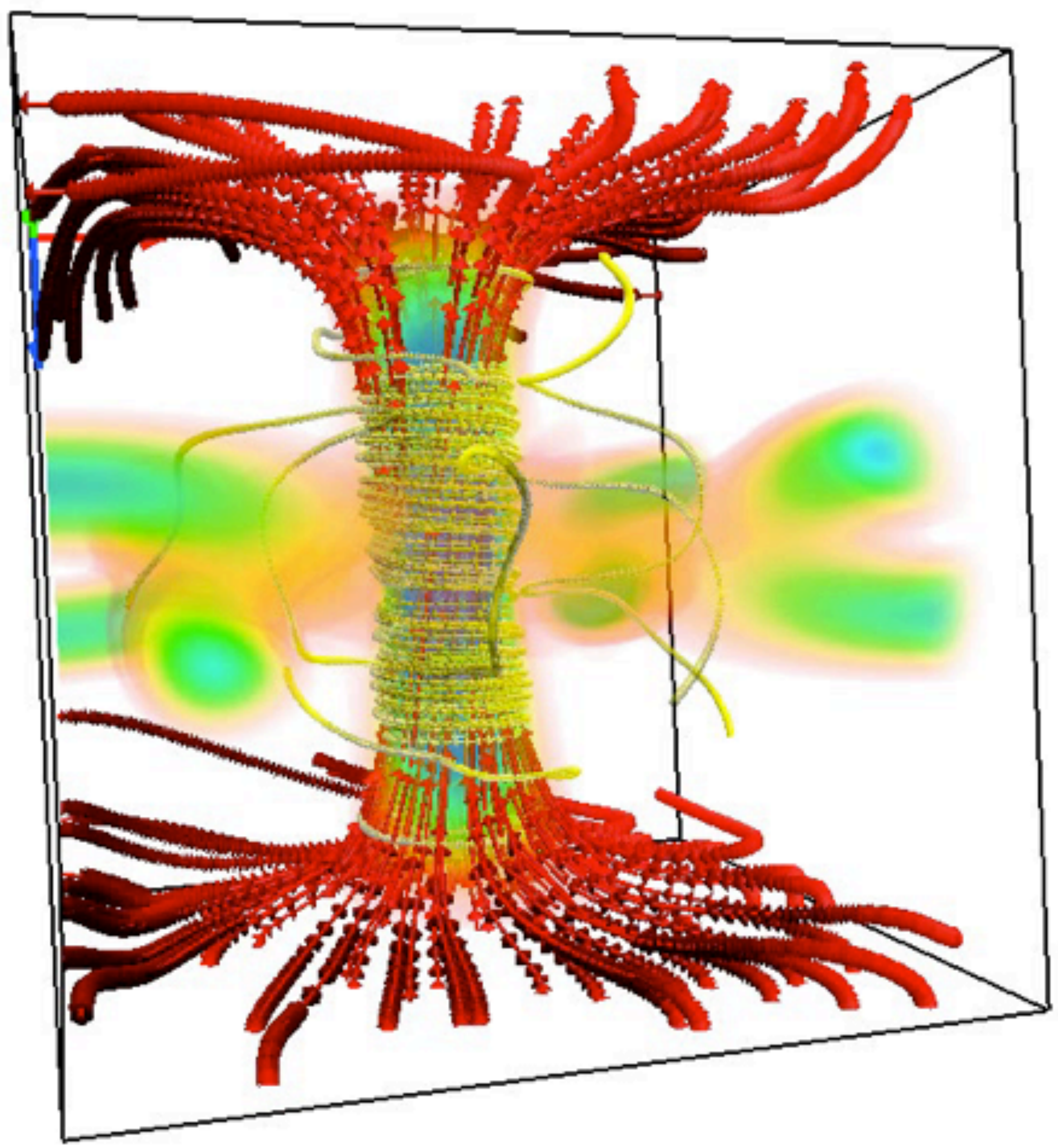


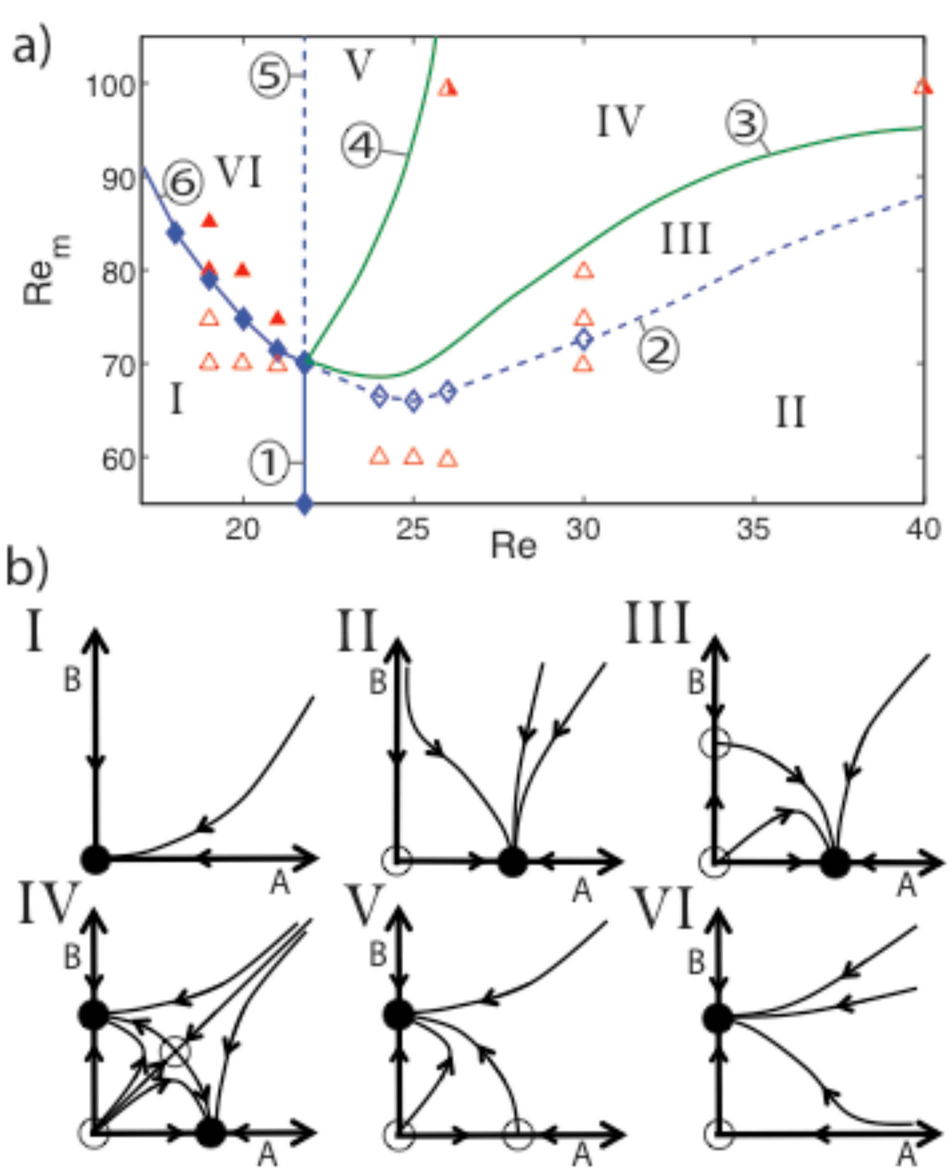
FIG. 8: (Color online) 3D visualizations (magnetic field in red, current in yellow and density plot of highest magnetic energy zones) of the growing modes: (a) ICC case, $\mathrm{Re}_m = 30$ (b) IIC case, $\mathrm{Re}_m = 80$, (c) CCC case, $\mathrm{Re}_m = 300$, (d) CCI case, $\mathrm{Re}_m = 300$.

III

Case	ICI	ICC	IIC	III	CCC	CCI
$\mathrm{Re}_{\mathrm{m}}^{\mathrm{c}}$	9	26	66	73	231	254



Codimension-2 bifurcation model



Denoting by $A(t)$ the real amplitude of the bifurcating velocity field $\mathbf{v}^{\rm PF}$ and $B(t)$ the real amplitude of the bifurcating magnetic field $\mathbf{b}$, we write coupled amplitude equations for A and B in the vicinity of the codimension-two bifurcation. The form of these equations is constrained by symmetry requirements, $A \to -A$ (pitchfork bifurcation of the velocity field) and $B \to -B$ ($\mathbf{b} \to -\mathbf{b}$ symmetry of the MHD equations).

Keeping the nonlinear terms to leading order, we get

$$\begin{aligned} \dot{A} &= \lambda A - \alpha A B^2 - A^3, \\ \dot{B} &= \mu B - \beta A^2 B - B^3. \end{aligned} \tag{29}$$

The coefficients of the cubic nonlinearities have been taken negative in order to get supercritical pitchfork bifurcations for the hydrodynamic instability in the absence of magnetic field ($B = 0$) and for the dynamo instability when $Re < Re^c$ and thus $A = 0$. The modulus of these coefficient can be taken equal to 1 by appropriate scalings of the amplitudes A and B. λ and μ are functions of Re and Re_m that vanishes at the codimension-two bifurcation point (Re^c, Re^c_m). To leading order, we have

Perspectives on Dynamo

- Rotations are bad and mirrors are good!
- Usual non-axial TYG growing mode obtained
- Axial dipoles also obtained (at higher Rmc) in case III
- Supercritical dynamo and velocity pitchfork
- Simple codim-2 model with bistability

Turbulence in TYG-symmetric MHD flows

several papers over the last few years…

Giorgio Krstulovic, Marc E. Brachet, and Annick Pouquet, Forced magnetohydrodynamic turbulence in three dimensions using Taylor-Green symmetries, Phys. Rev. E 89, 043017 (2014)

Marc Brachet, Miguel Bustamante, Giorgio Krstulovic, Pablo Mininni, Annick Pouquet and Duane Rosenberg, Ideal evolution of magnetohydrodynamic turbulence when imposing Taylor-Green symmetries, Phys. Rev. E 87, 013110 (2013)

Pouquet A., Lee E., Brachet ME., Mininni PD., and Rosenberg D., The dynamics of unforced turbulence at high Reynolds number for Taylor-Green vortices generalized to MHD, Geophysical & Astrophysical Fluid Dynamics, 104: 2 (2010)

Lee E., Brachet ME., Pouquet A., Mininni PD., and Rosenberg D., Paradigmatic flow for small-scale magnetohydrodynamics: Properties of the ideal case and the collision of current sheets, Phys. Rev. E 78, 066401 (2008)

Symmetries are not broken in these problems…

- In turbulent decay runs TYG symmetries are not broken
- Using a code that implements the symmetries gains a large factor in resolution
- Will now explain method and show a few results

Numerical method

The simulations reported in this paper were performed using a special purpose symmetric parallel code developed from that described in [13, 14]. The workload for a timestep is (roughly) twice that of a general periodic code running at a quarter of the resolution. Specifically, at a given computational cost, the ratio of the largest to the smallest scale available to a computation with enforced Taylor-Green symmetries is enhanced by a factor of 4 in linear resolution. This leads to a factor of 32 savings in total computational time and memory usage. The code is based on FFTW and a hybrid MPI-OpenMP scheme derived from that described in [15]. The runs were performed on the IDRIS BlueGene/P machine. At resolution 4096^3 we used 512 MPI processes, each process spawning 4 OpenMP threads.

Symmetries are not broken!!!

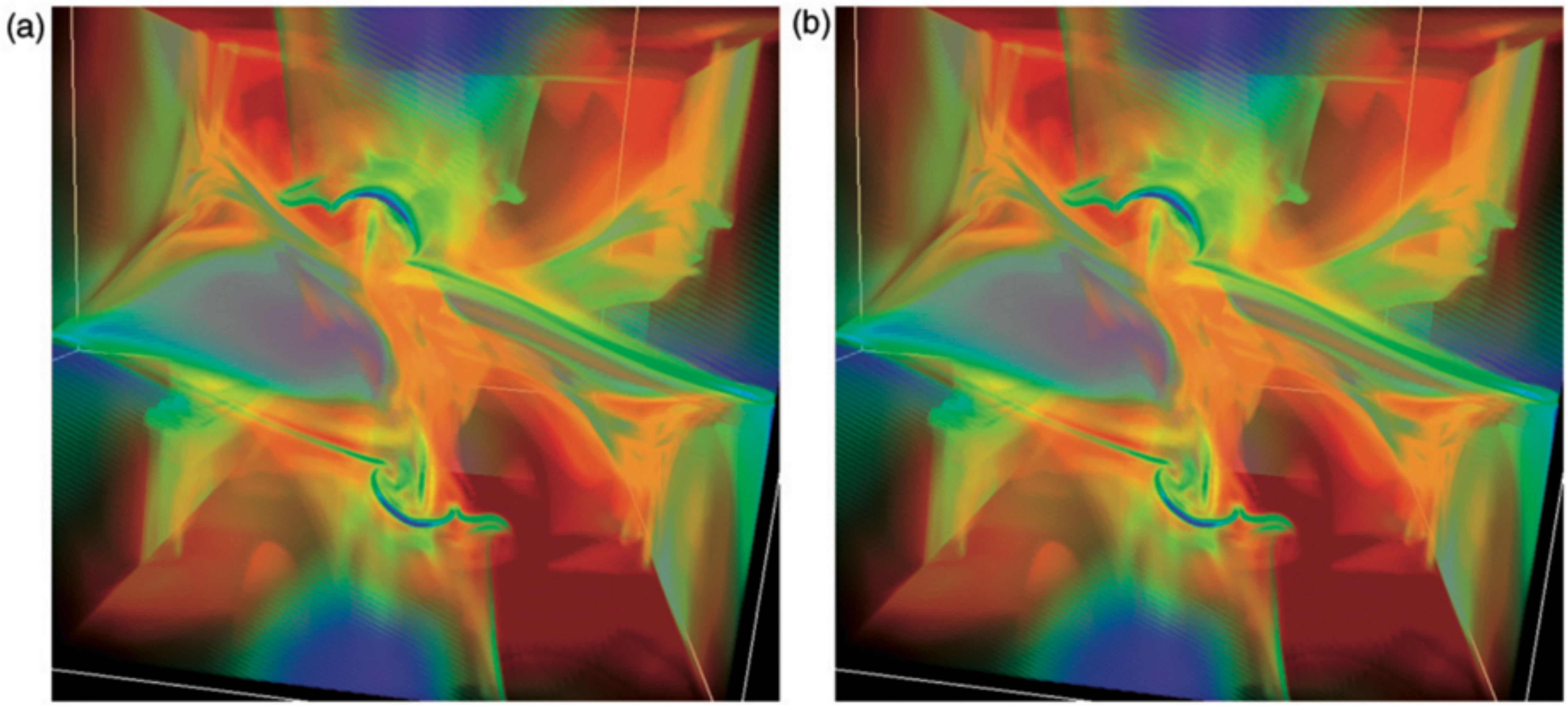
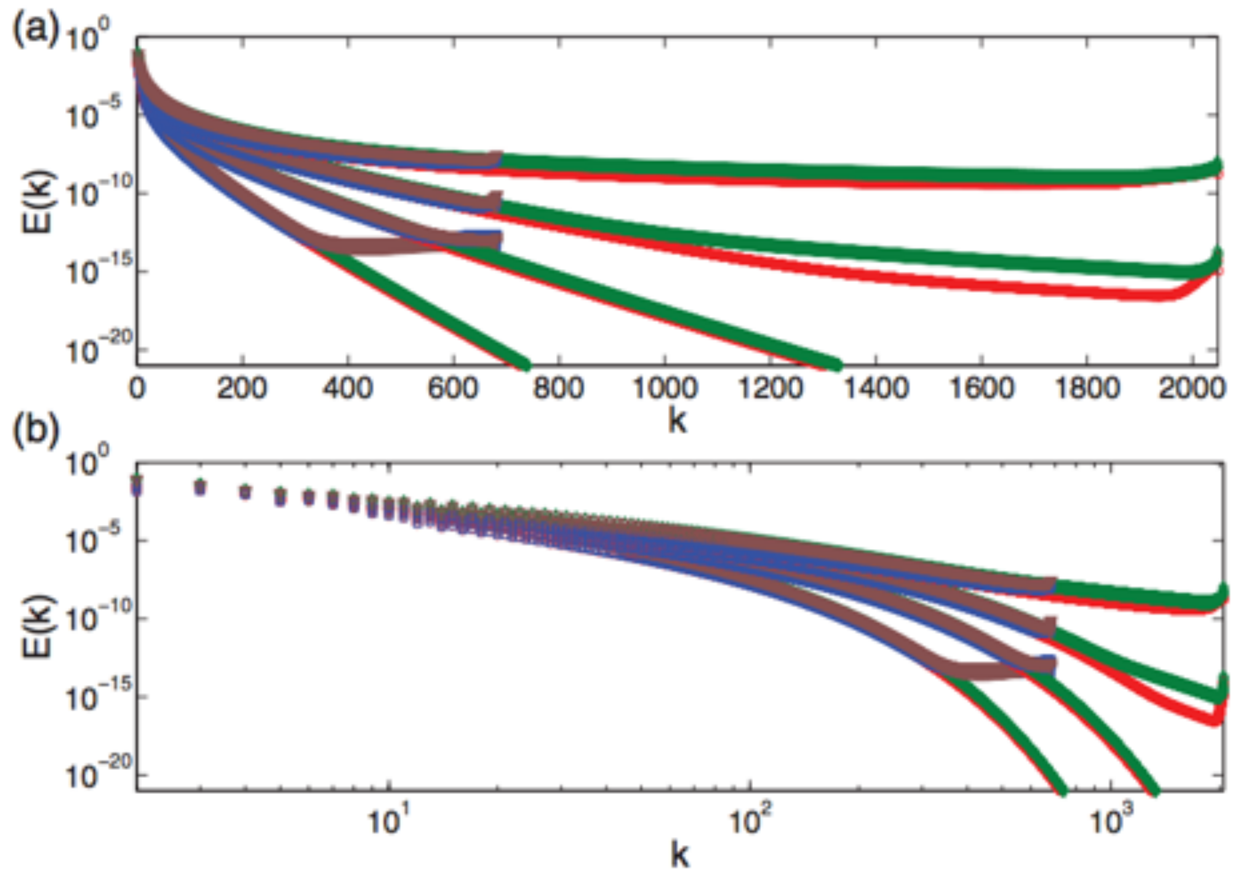


Figure 2. Visualization (VAPOR software, Clyne *et al*. 2007) of the current density inside the $[0, \pi]^3$ box for runs on grids of 512^3 points; full DNS (a) and code with symmetries (b). There is also a strong current sheet on the walls (not shown).

Singularity problem



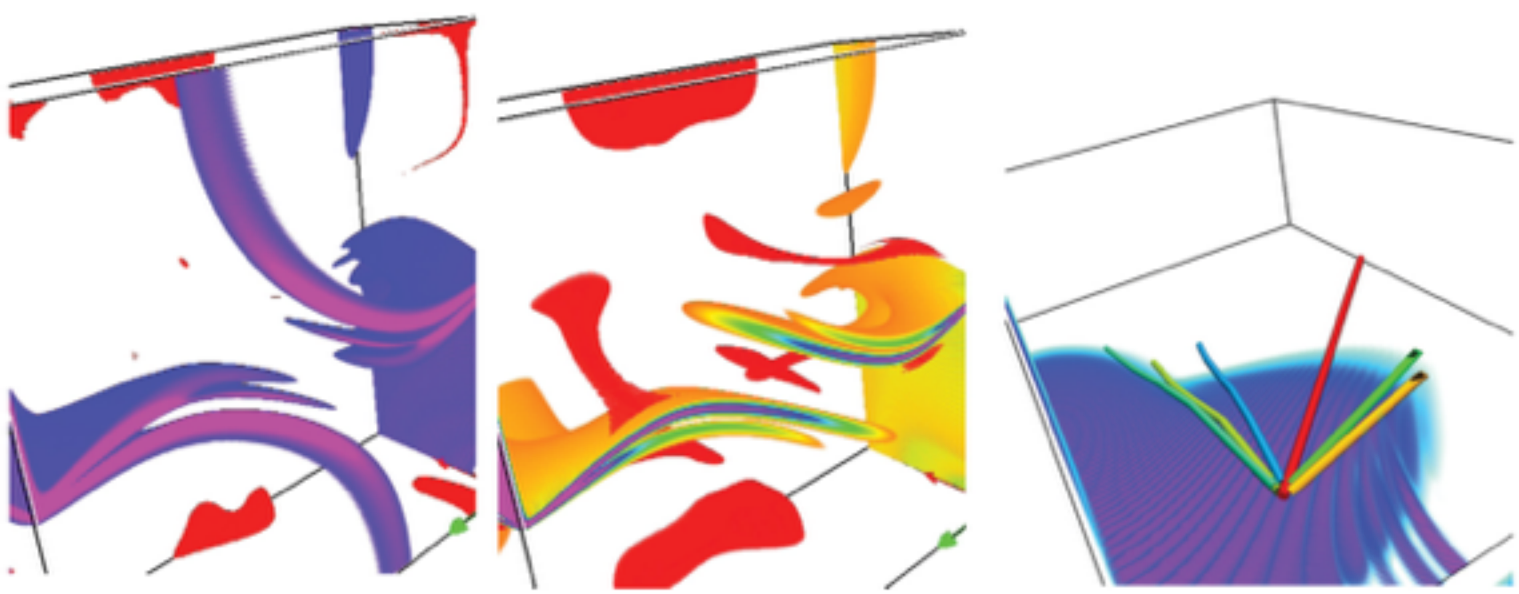


FIG. 10. (Color online) Perspective volume rendering using the VAPOR software [54,55] of the vorticity (left) and of the current density (middle) at $t = 2.54$. Note the occurrence of a double layer structure due to the collision and subsequent joining of two sheets. At a later time ($t = 2.65$; right), the magnetic field lines taken on these two colliding sheets all go to the same location, which coincides with the maximum of the current (coordinates given in the table), implying sharp localized bending (and possibly torsion) of magnetic field lines in the vicinity of that maximum.

Forced MHD turbulence

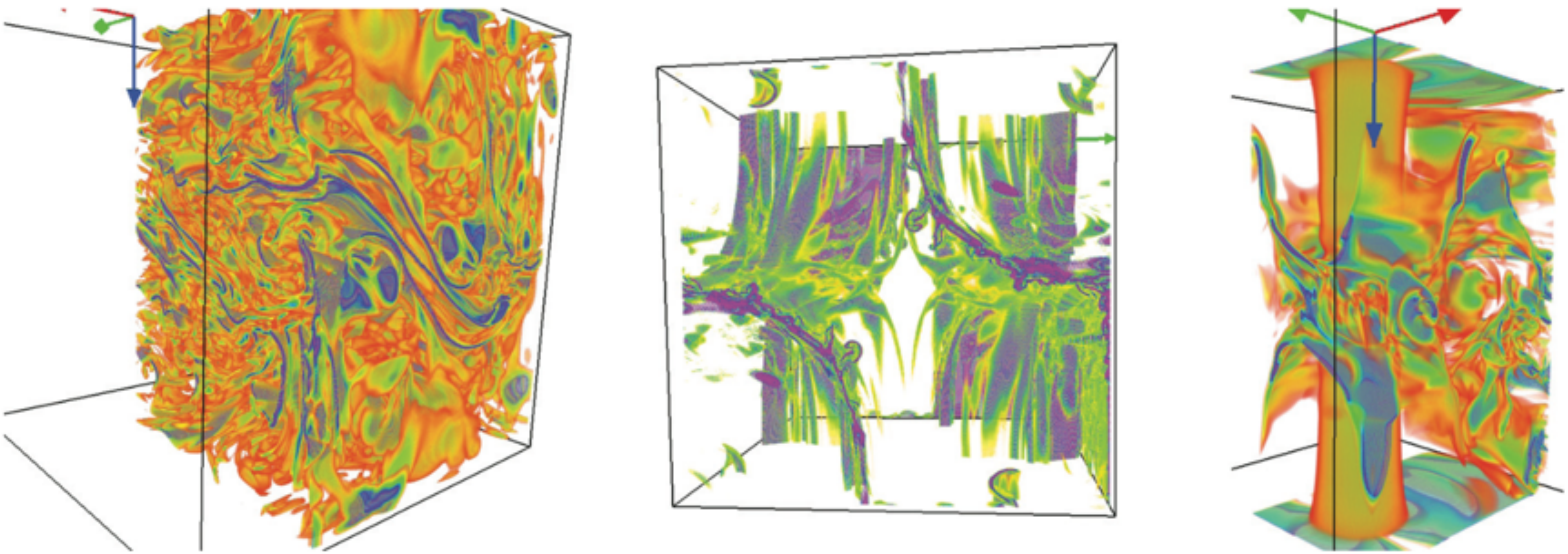


FIG. 8. (Color online) Visualizations of the magnetic energy at the end of the runs. Left: run $C2$. Center: run I. Right: run A.

Discussion/conclusion

- Large scale symmetries are useful
- They can be used to specify types of boundaries
- They can also be used to maximize the ratio of large to small scale
- After all periodicity itself is a symmetry of the large scales!
- These points should be taken into account when planning a long DNS…