A review on wavelet transforms and their applications to MHD and plasma turbulence I

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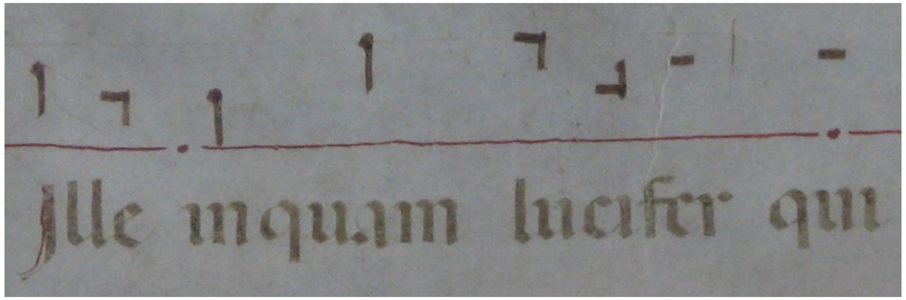
Methods for Analyzing Turbulence Data
Meudon, 29 May 2015

Choice of the appropriate representation

'It could be interesting, in communication theory, to represent an oscillating signal by a superposition of elementary wavelets, each of them having both a frequency and a time localization quite well defined. The useful information is indeed often carried by both the emitted frequencies and by the time structure of the signal (music is a characteristic example of that). The representation of a signal as function of time cannot exhibit the frequency content, while in contrast its Fourier analysis hides the time of emission and the duration of each elements of the signal. An adequate representation should combine the advantages of these two complementary descriptions, while providing a discrete character appropriate to the theory of communication.'

Roger Balian
Un principe d'incertitude
fort en théorie du signal
CRAS, 292, II (1981)

Representation for music



Guido d'Arezzo
Micrologos
1025

7 tones:
ut re mi fa sol la si
sa re ga ma pa da ni

12 half tones:

$$f_n = f_0 \cdot a^n$$
$$a = 2^{1/12}$$

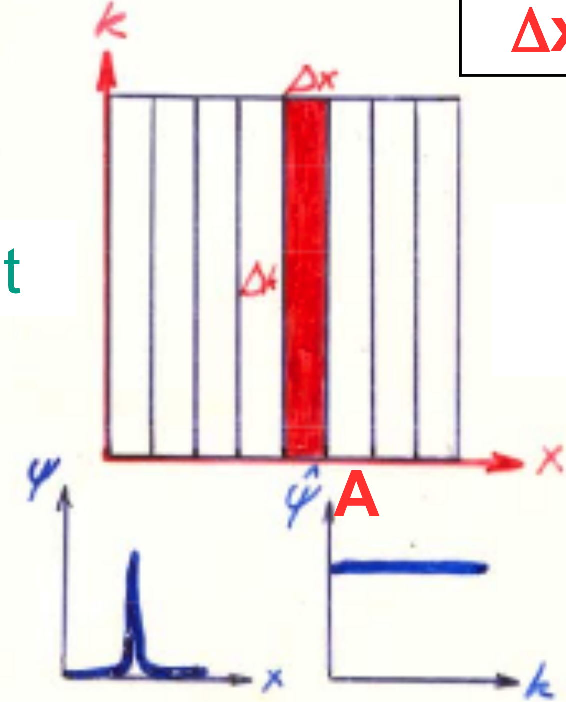
Integral transforms

Analysis $\qquad T_f(k) = \int f(\vec{x})\, \psi_k(\vec{x})\, d\vec{x}$

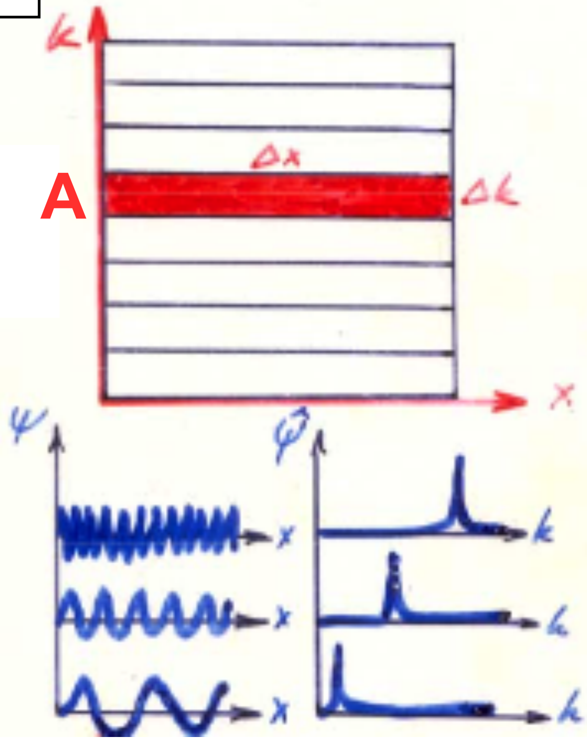
Synthesis : $\qquad f(\vec{x}) = \frac{1}{c} \int T_f(k)\, \psi_x(\vec{k})\, d\mu(k)$

Δx Δk = A

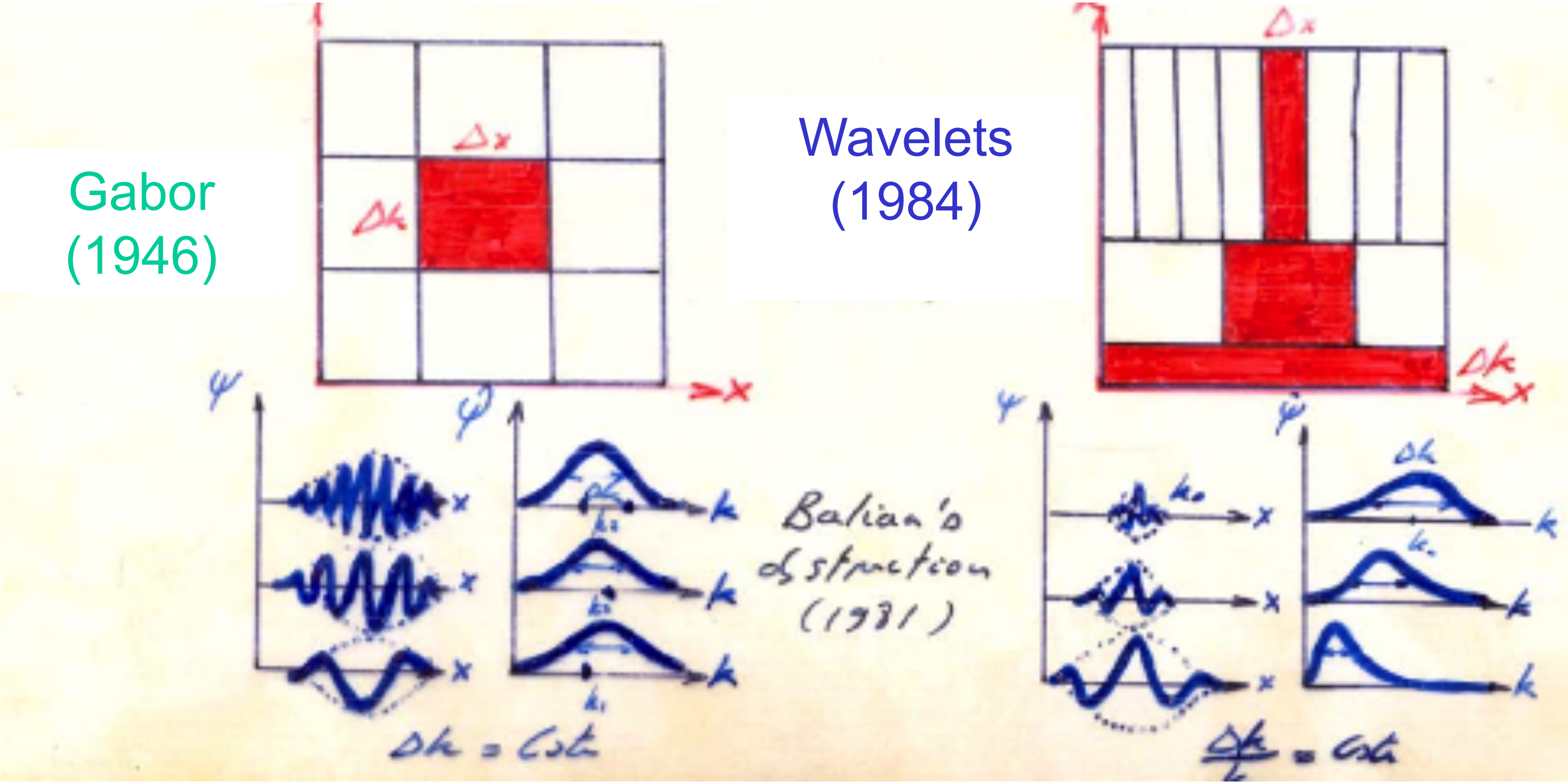
Gridpoint



Fourier
(1807)



Optimal phase space tiling



Space-wavenumber representation

$\Delta x \, \Delta k$ = information atom

Space-scale representation

Choice of the analyzing wavelet

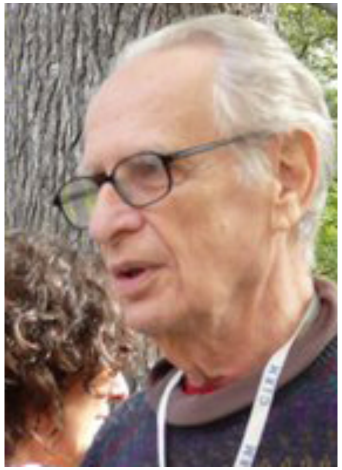
Admissibility condition

$$C_\psi = \int_0^\infty \left|\widehat{\psi}(k)\right|^2 \frac{dk}{|k|} < \infty$$

$$\int_{-\infty}^{\infty} \psi(x)\,dx = 0 \qquad \text{or} \quad \widehat{\psi}(0) = 0$$

Jean Morlet

Alex Grossmann



Analyzing wavelet family generated by translation (b) and dilation (a)

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{x-b}{a}\right)$$

Grossmann and Morlet,
Decomposition of Hardy functions into square integrable wavelets of constant shape,
*SIAM J. math. Anal., **15**(4), 723-736, 1984*

Continuous wavelet transform (CWT)

Analysis

$$\widetilde{f}(a,b) = \int_{-\infty}^{\infty} f(x)\psi^*_{a,b}(x)\,dx$$

Synthesis

$$f(x) = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^{\infty} \widetilde{f}(a,b)\,\psi_{a,b}(x)\,\frac{da\,db}{a^2}$$

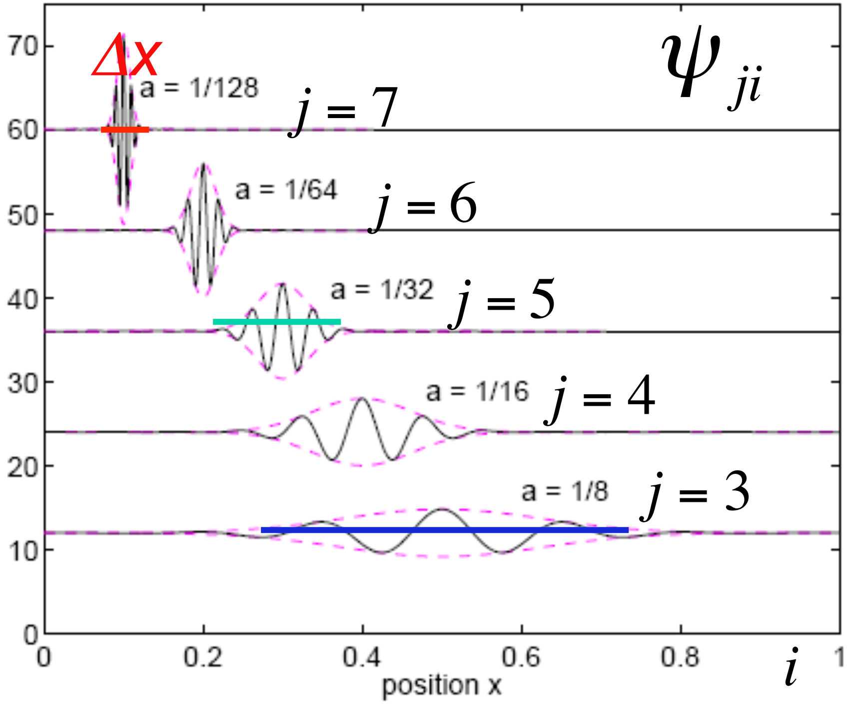
Parseval's identity

$$\langle f_1, f_2 \rangle = \int_{-\infty}^{\infty} f_1(x)\,f_2^*(x)\,dx = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^{\infty} \widetilde{f}_1(a,b)\,\widetilde{f}_2{}^*(a,b)\,\frac{dadb}{a^2}$$

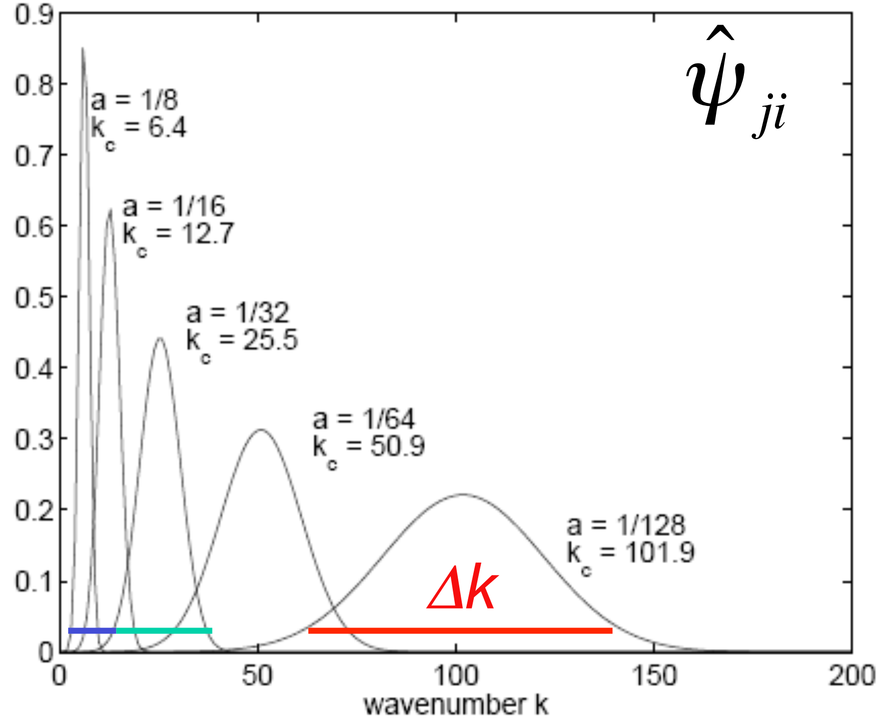
Wavelet representation

$\Delta x \, \Delta k > C$

Physical space



Spectral space

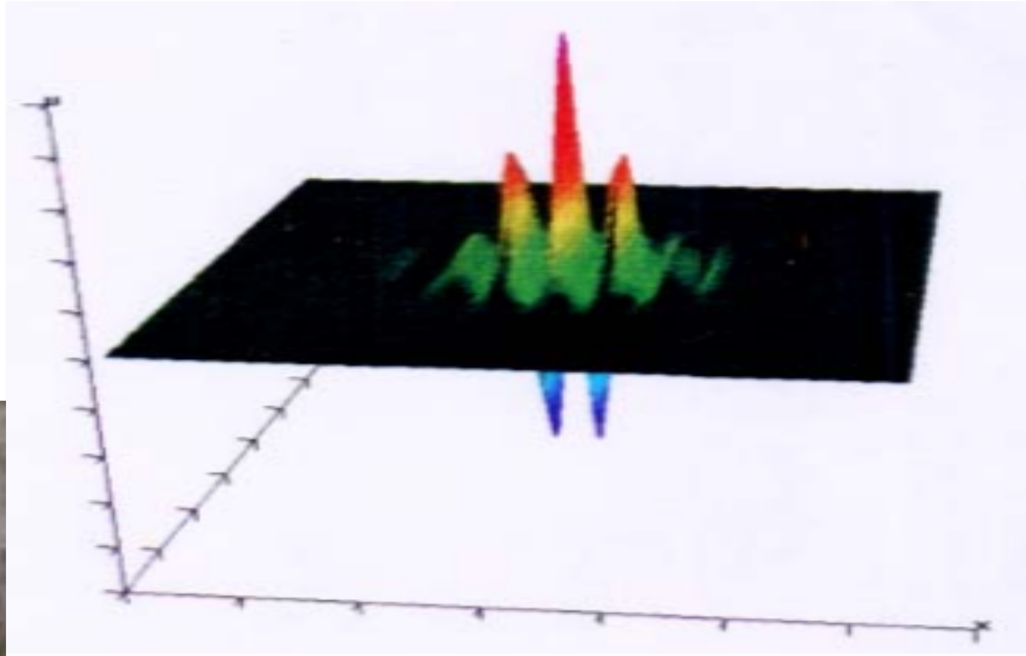


Farge,
Wavelet transforms and
their applications to turbulence
Ann. Rev. Fluid Mech., ***92****, 1992*

Farge and Schneider,
Wavelets: application to turbulence,
Encyclopedia of Mathematical Physics,
Springer, 408-42, 2006

2D continuous wavelet transform

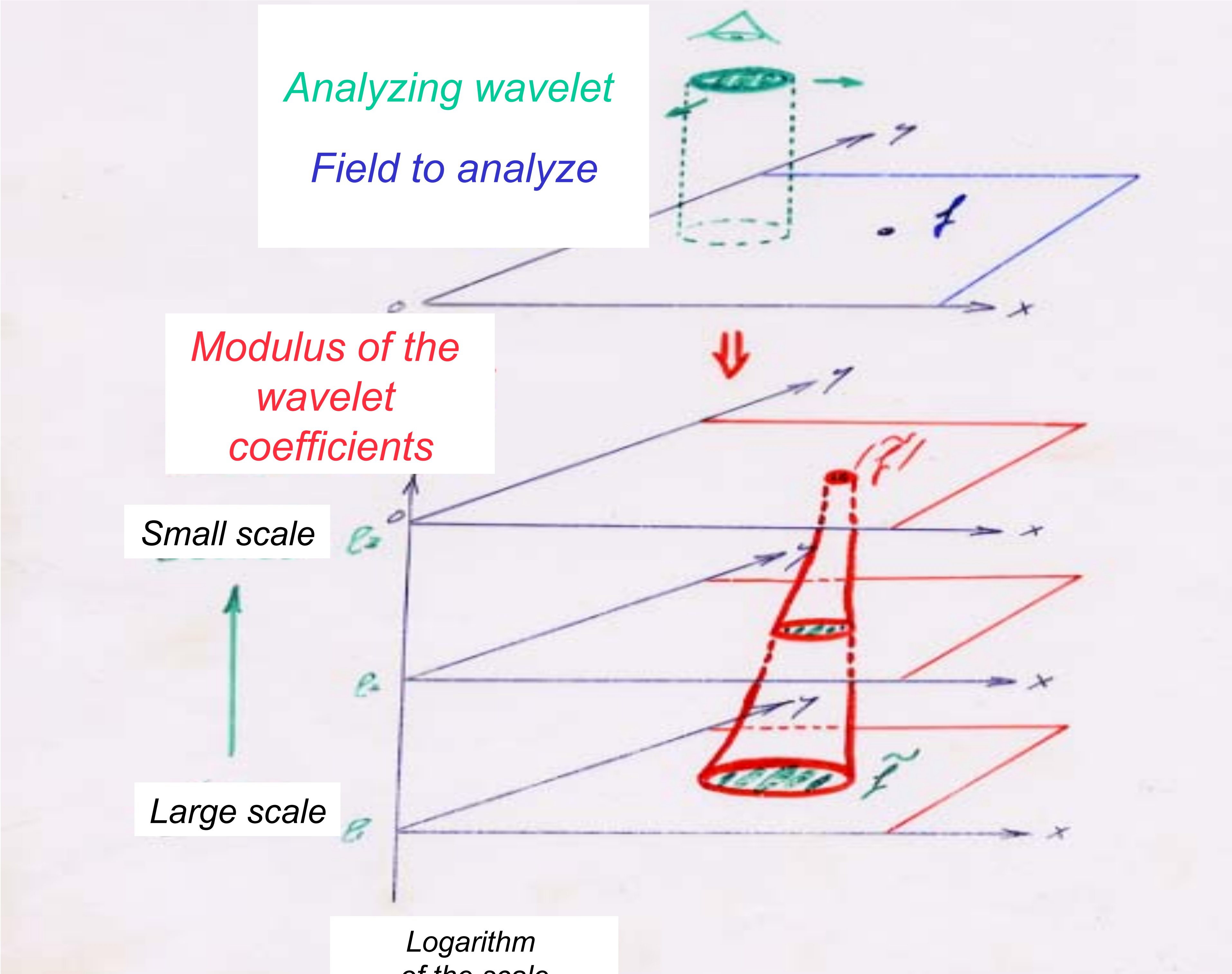
Jean-Pierre Antoine



2D Morlet mother wavelet

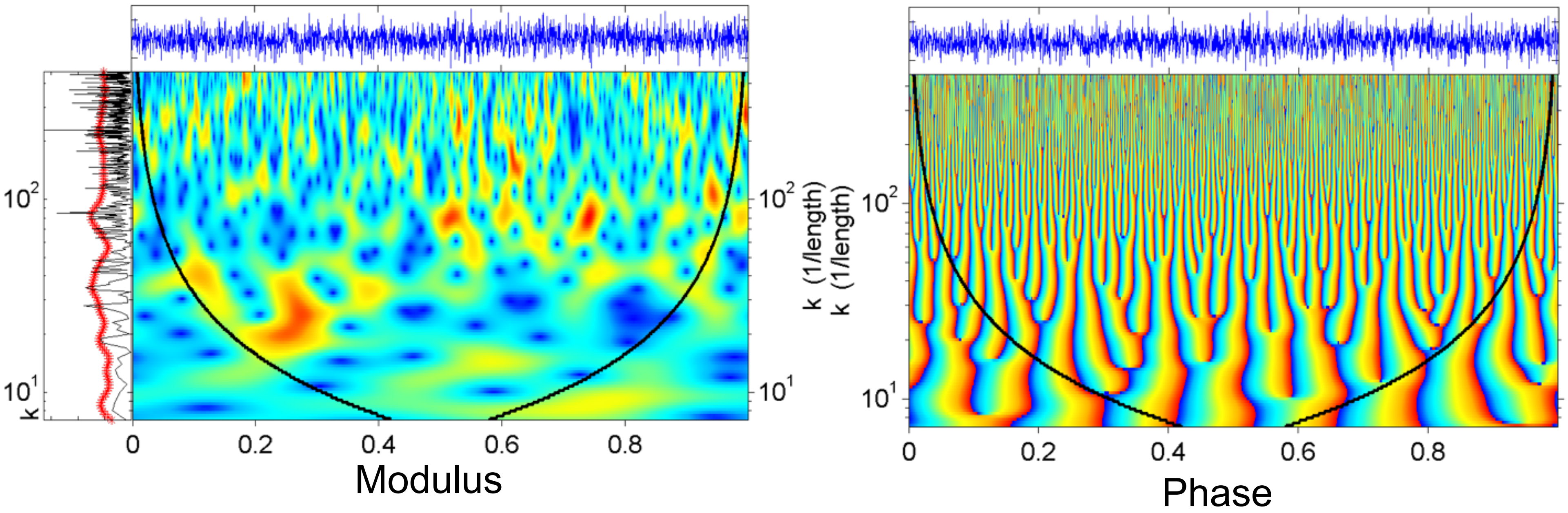
Romain Murenzi

The wavelet family is generated by translating, dilating and rotating the 2D mother wavelet


Analyzing wavelet
Field to analyze
Modulus of the wavelet coefficients
Small scale
Large scale
Logarithm

Reproducing kernel of the CWT

The CWT of a Gaussian white noise reveals its reproducing kernel

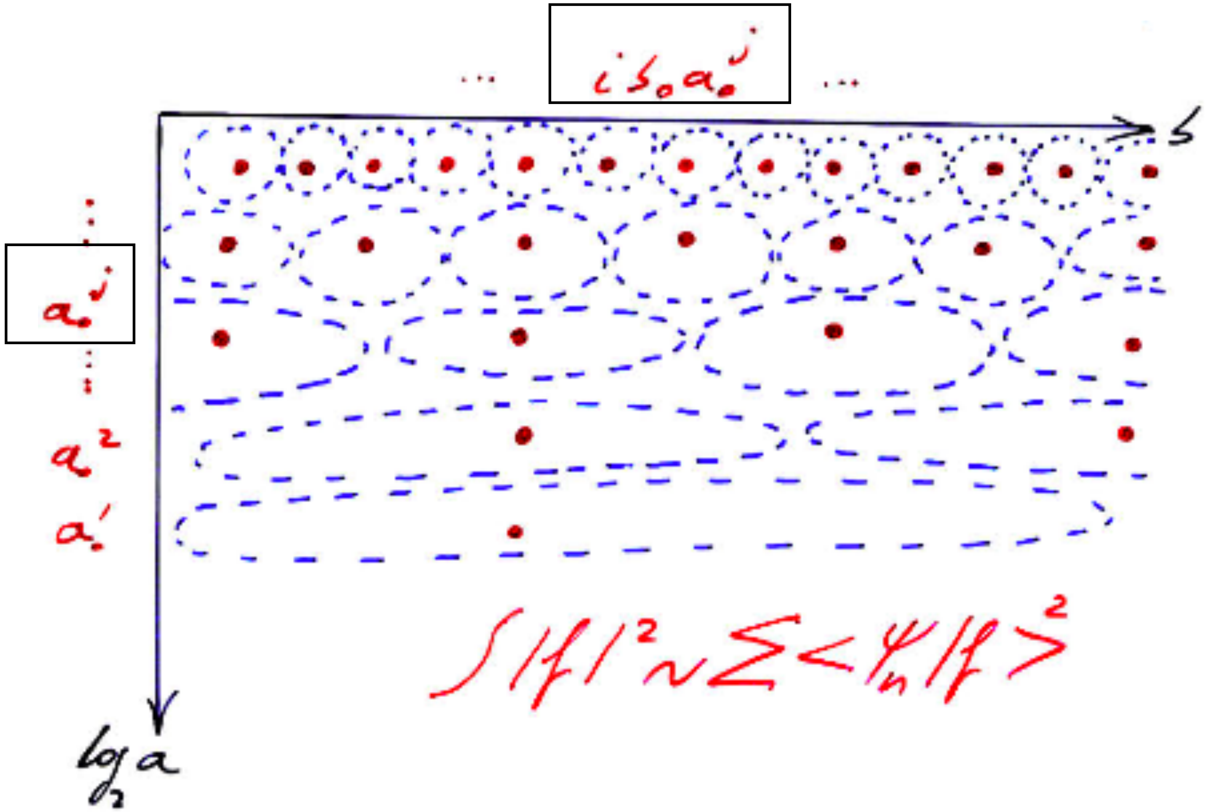


The is the correlation between the wavelets which corresponds to the redundancy between the coefficients

$$K(b', a', b, a) = \langle \psi_{b'a'} | \psi_{ba} \rangle$$

Wavelet frame

We can then select a finite number of wavelets restricted to a discrete grid optimally chosen such that the wavelet family associated to this grid constitutes a quasi-orthogonal basis $\Longrightarrow$ *a wavelet frame*



For example

for Marr wavelet

we need

$a_0 = 2^{1/2}$

$b_0 = 1/2$

Orthogonal wavelet transform

Wavelet analysis :

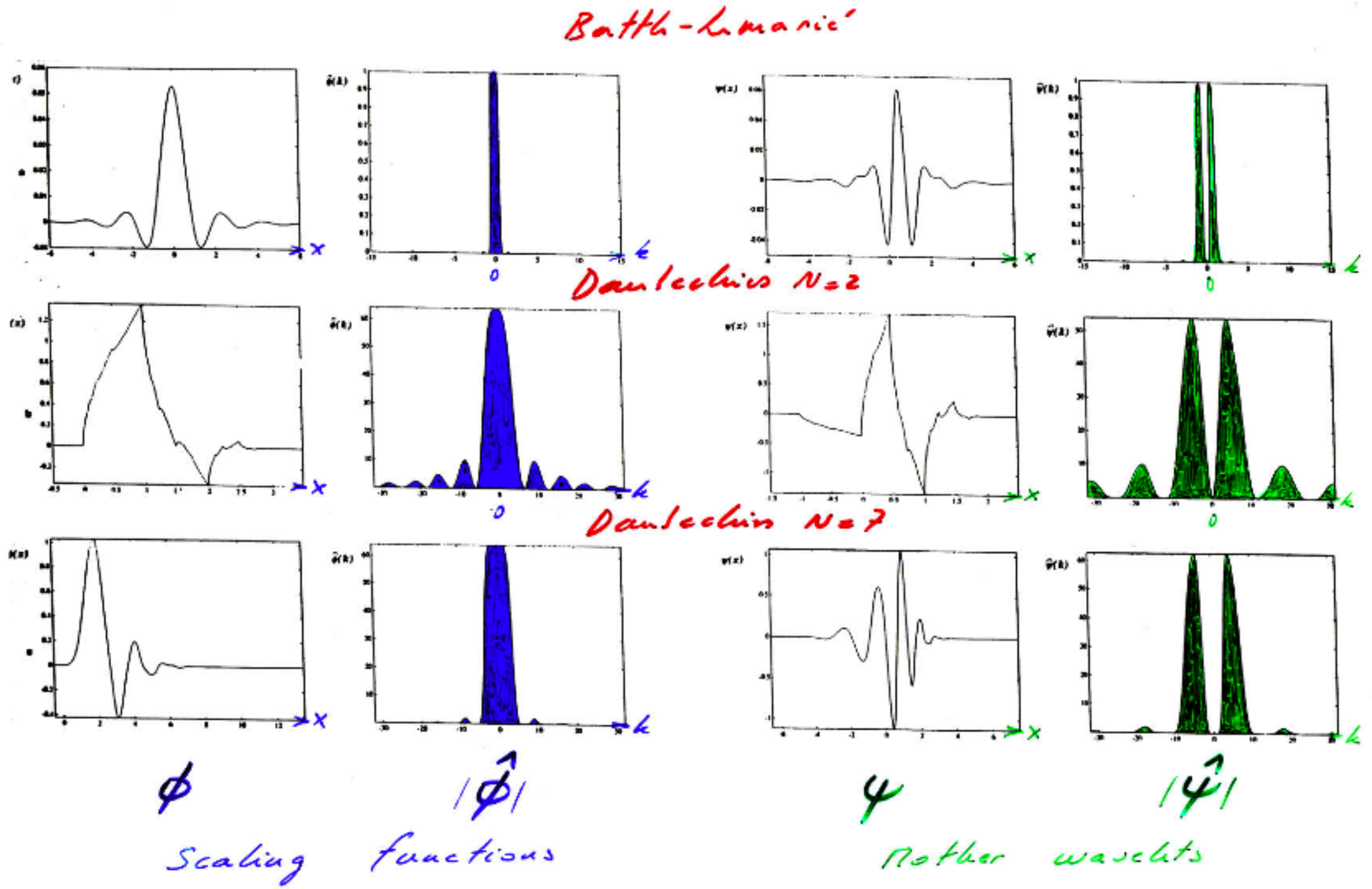
$$\widetilde{f}_{ji} = \langle \psi_{ji} | f \rangle \quad \text{with} \quad \psi_{ji} = 2^{j/2} \psi(2^j x - i)$$

Wavelet synthesis :

$$f = \sum_{ji} \langle \psi_{ji} | f \rangle \psi_{ji}$$

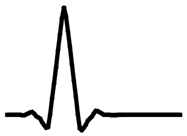
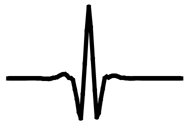
A signal sampled on N points is
wavelet analyzed and synthetized in CN operations
if one uses compactly-supported wavelets
computed from a quadratic mirror filter of length M.

Examples of orthogonal wavelets



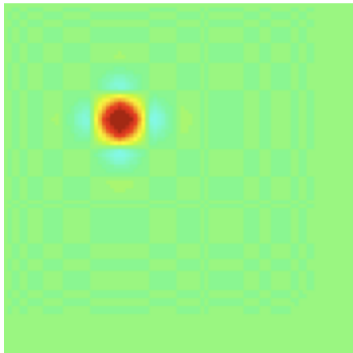
2D orthogonal wavelets

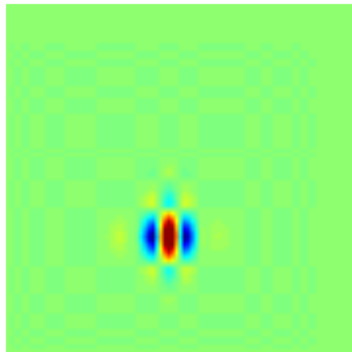
Scaling function *Wavelet*

ϕ  ψ 

$\phi(x)\phi(y)$

Coarse approximation



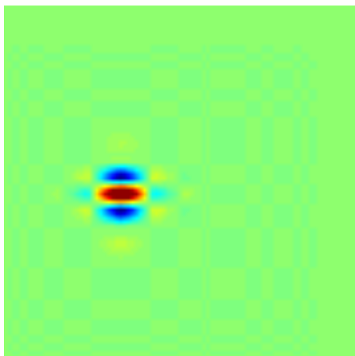


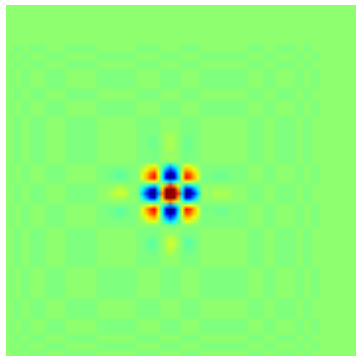
$\psi(x)\phi(y)$

Horizontal details

$\phi(x)\psi(y)$

Vertical details





$\psi(x)\psi(y)$

Diagonal details

3D orthogonal wavelets

A 3D vector field v(x) sampled on $N = 2^{3J}$ equidistant grid points

$\psi_\lambda(\boldsymbol{x})$ 3D wavelet $\longrightarrow$ orthogonal wavelet series

$$\boldsymbol{v}(\boldsymbol{x}) = \sum_{\lambda \in \Lambda} \widetilde{\boldsymbol{v}}_\lambda \psi_\lambda(\boldsymbol{x}), \quad \widetilde{\boldsymbol{v}}_\lambda = \langle \boldsymbol{v}, \psi_\lambda \rangle$$

$$\Lambda = \{\lambda = (j, i_n, \mu, \ldots, J-1,\ i_n = 0, \ldots, 2^j - 1, n = 1, 2, 3, \text{ and } \mu = 1, \ldots, 7\}$$

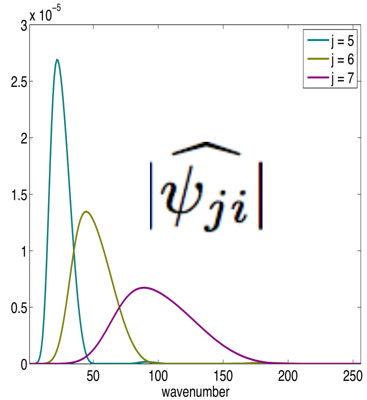
$N_j = 7 \times 2^{3j}$, wavelet coefficients at a scale indexed by j

- fast algorithm with linear complexity
- no redundancy between the coefficients

We use Coifman 12 wavelet
compactly supported with four vanishing moments.

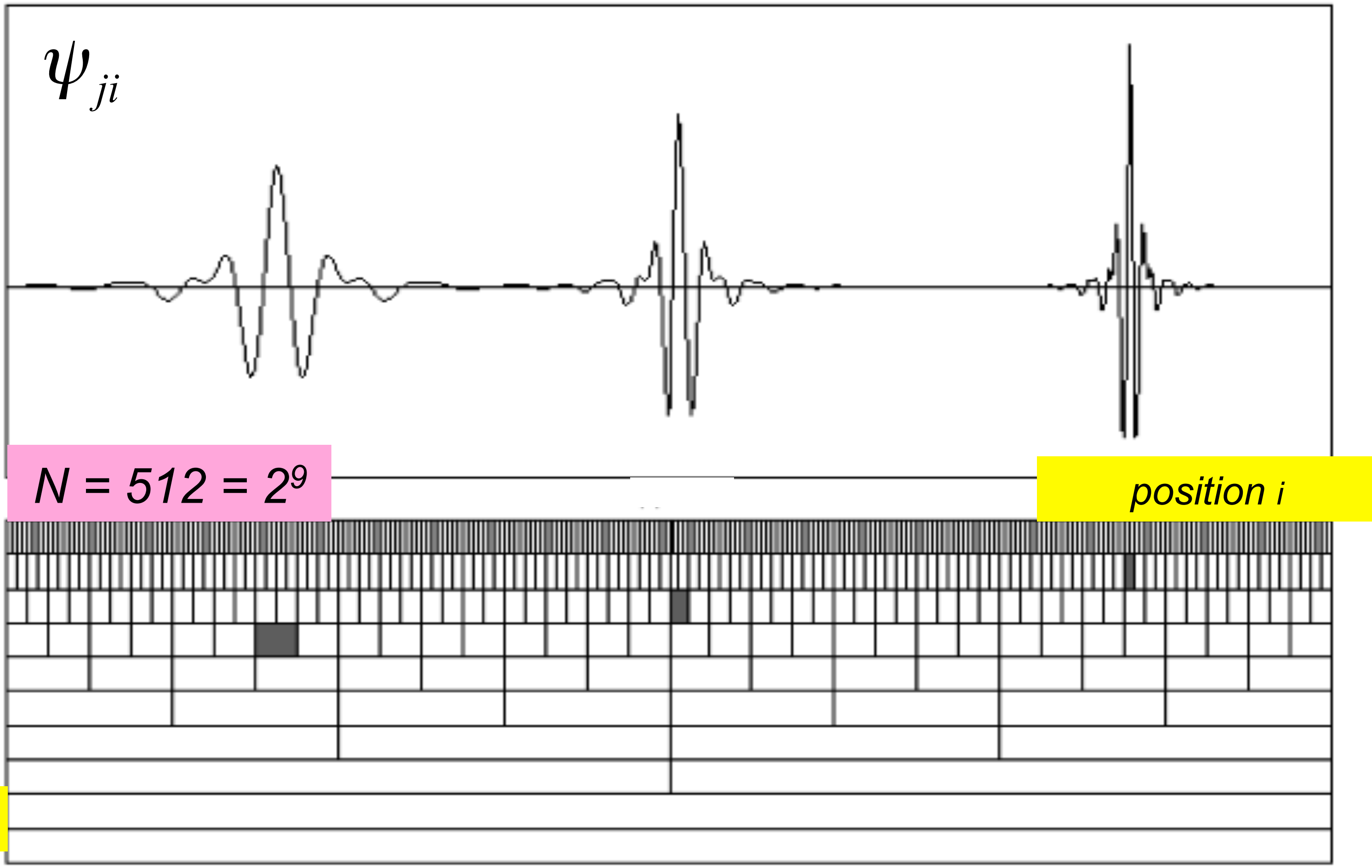
Orthogonal wavelet representation

Wavelets



Wavelet coefficients

$$\tilde{f}_{ji} = \langle \psi_{ji} | f \rangle$$

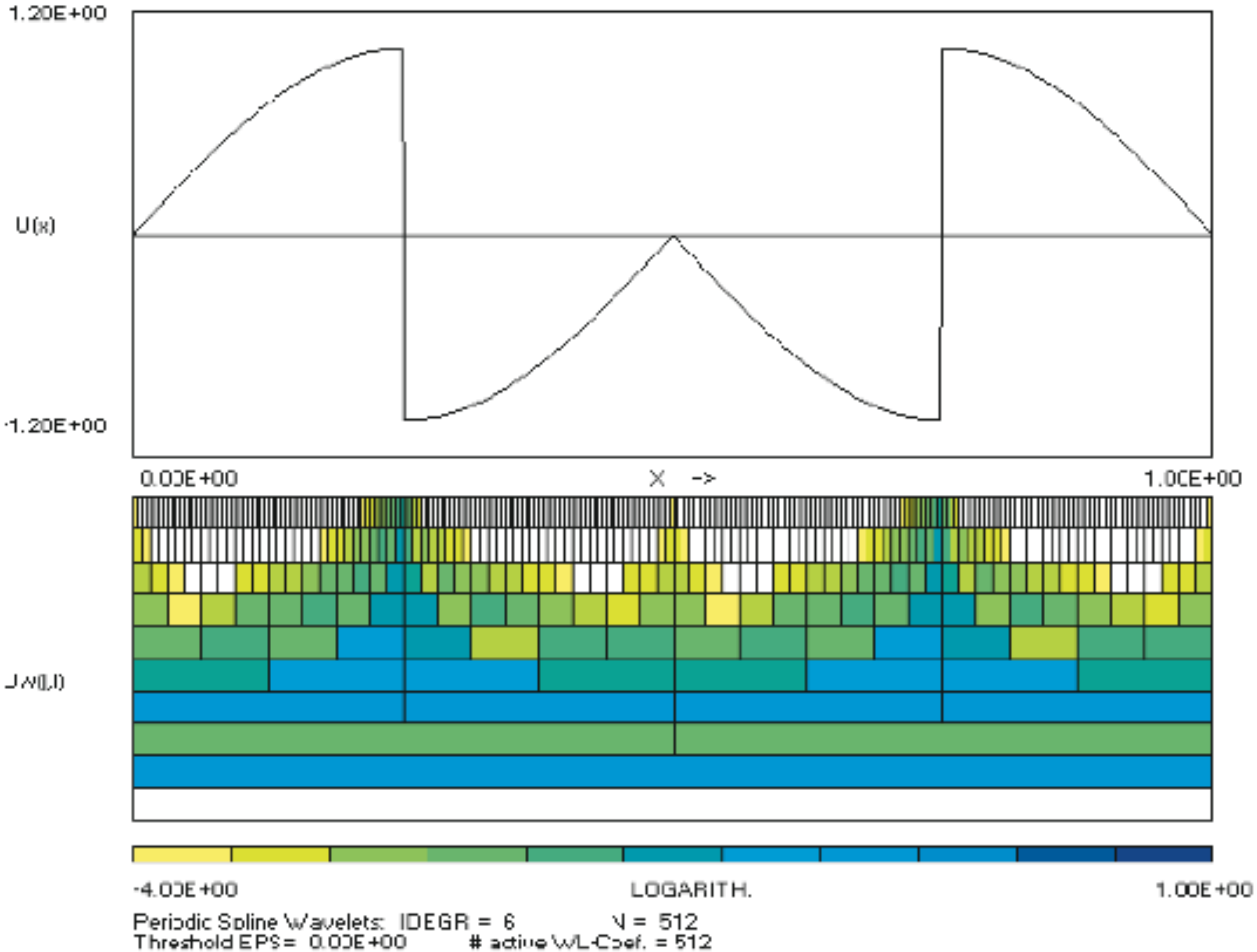


Mallat,
A wavelet tour of
signal processing, 3rd edition,
Academic Press, 2008

Academic example

Function sampled on N = 512 grid-points

N = 512 wavelet coefficients

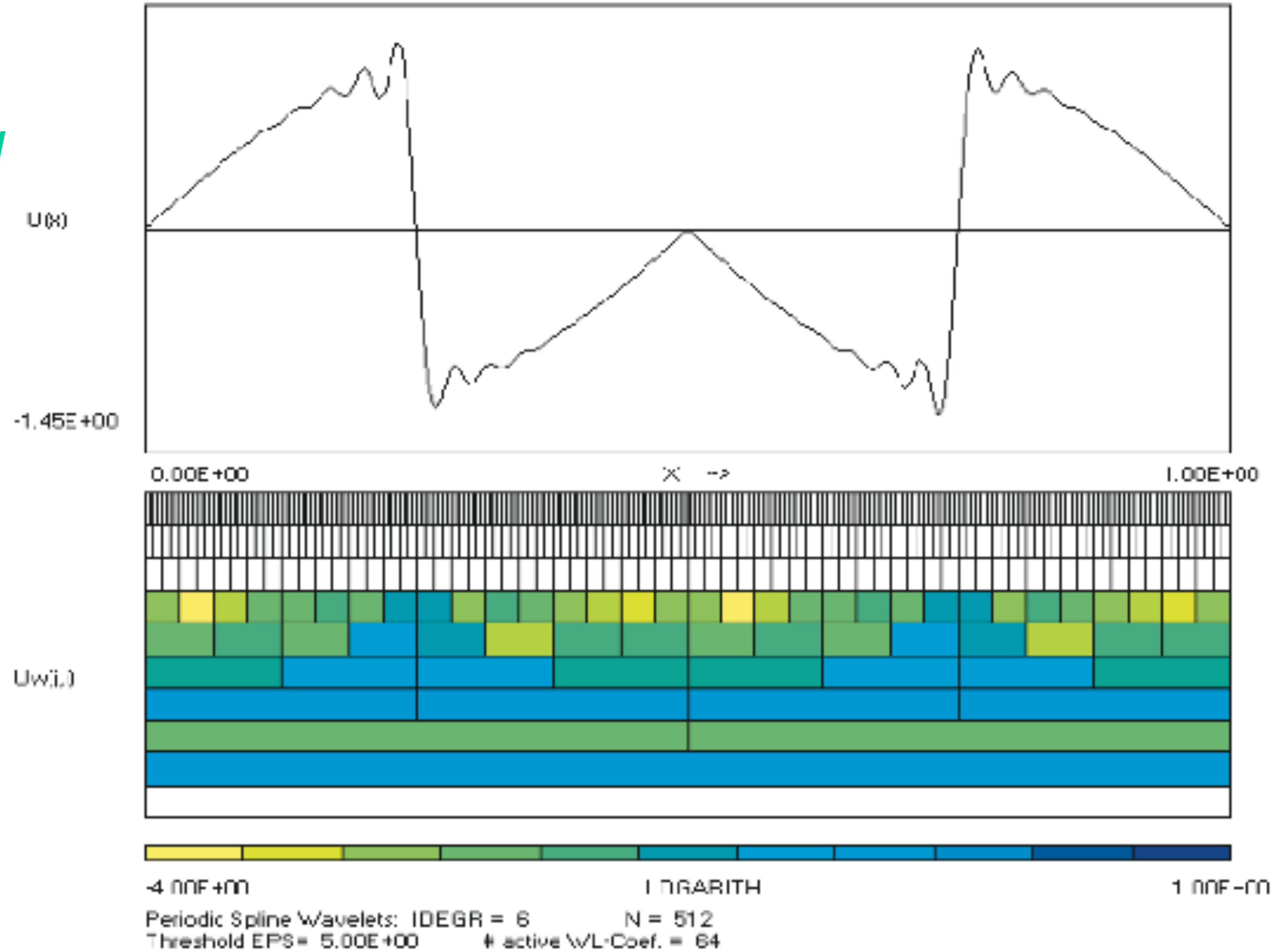


Linear approximation

Function reconstructed from 64 wavelet coefficients

64 wavelet coefficients such that

$j \leq 6$

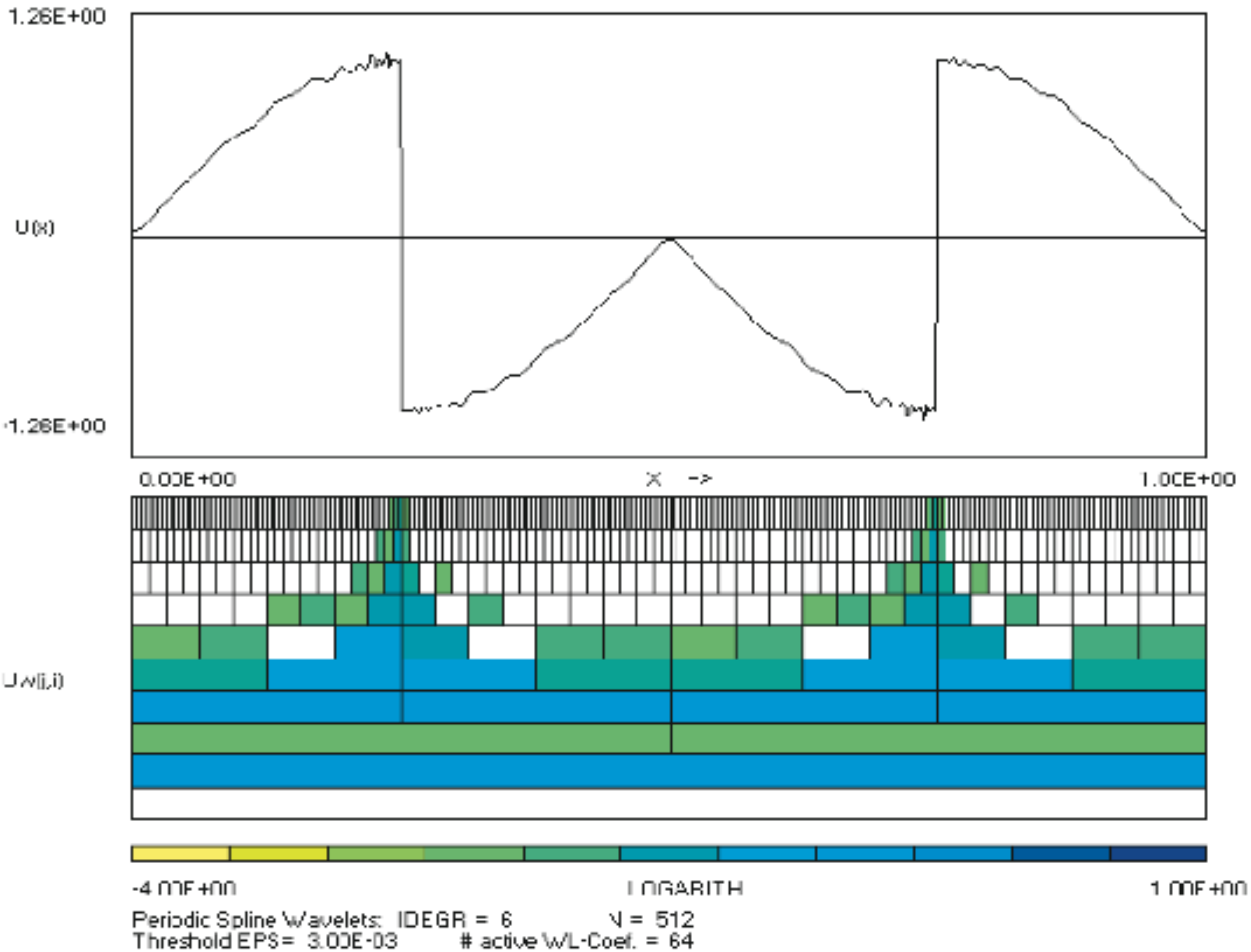


Nonlinear approximation

Function reconstructed from 64 wavelet coefficients

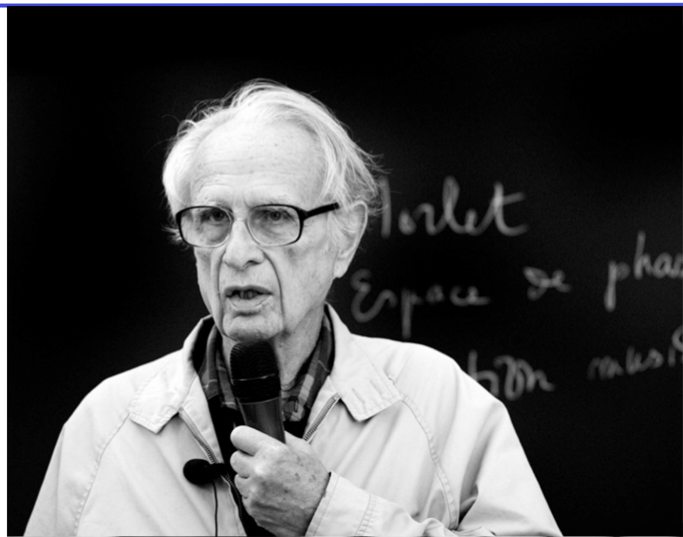
64 wavelet coefficients such that

$$|\widetilde{f}_{ji}| > \epsilon$$



Wavelet analysis

Grossmann



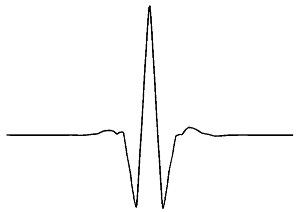
Meyer

Daubechies

Mallat

Continuous / orthogonal wavelets

Analyzing functions are
translates and dilates
of an oscillating function (of zero mean)



Well localized in both space and wavenumber

$$\tilde{f}(l, \vec{x}) = \langle \psi_{l,\vec{x}} | f \rangle$$

Continuous wavelets	Orthogonal wavelets
$\psi_{l,\vec{x}}(x') = \frac{1}{l^{n/2}} \psi(\frac{\vec{x}' - \vec{x}}{l})$	$\psi_{j,i}(x') = 2^{j/2} \psi(2^j x' - i)$
• Translates and dilates vary continuously • Redundant representation	• Translates and dilates are on a discrete dyadic grid • Orthogonal basis
• Coefficients are easy to read • Unfold in both space and scale • Good for analysis	• Coefficients not easy to read • sampled on a dyadic grid • For filtering and compression

How to extract coherent structures?

Since there is not yet a universal definition of coherent structures which emerge out of turbulent fluctuations due to the nonlinear interactions, we adopt an apophetic method : instead of defining what they are, we define what they are not.

For this we propose the minimal statement :
'Coherent structures are not noise'

$\Longrightarrow$

Extracting coherent structures becomes a denoising problem, not requiring any hypotheses on the structures themselves but only on the noise to be eliminated.

Choosing the simplest hypothesis as a first guess, we suppose we want to eliminate an additive Gaussian white noise, and for this we use a nonlinear wavelet filtering.

*Farge, Schneider et al., Phys. Fluids, **15** (10), 2003*

*Azzalini, Farge, Schneider, ACHA, **18** (2), 2005*

Denoising using wavelets

Gaussian **white noise** is by definition **equidistributed** among all modes and the amplitude of the coefficients is given by its r.m.s., whatever the functional basis one considers.

Therefore the coefficients of a noisy signal whose amplitudes are larger than the r.m.s. of the noise belong to the denoised signal. This procedure corresponds to **nonlinear filtering**.

The advantage of performing such a nonlinear filtering using the wavelet representation is that the **wavelet coefficients** preserve the space locality, since wavelets are functions localized in both physical and spectral space.

Since we do not know *a priori* the r.m.s. of the noise, we have proposed an iterative procedure which takes as first guess the r.m.s. of the noisy signal.

Azzalini, M. F., Schneider, 2005
Appl. Comput. Harmonic Analysis, ***18*** *(2)*

Wavelet denoising algorithm

Apophatic method :

- *no hypothesis on the structures,*
- *only hypothesis on the noise,*
- *simplest hypothesis as our first choice.*

Hypothesis on the noise :

$$f_n = f_d + n$$

n *Gaussian white noise,*
$<f_n^2>$ *variance of the noisy signal,*
N *number of coefficients of* f_n.

Wavelet decomposition :

$$\tilde{f}_{ji} = < f \mid \psi_{ji} >$$

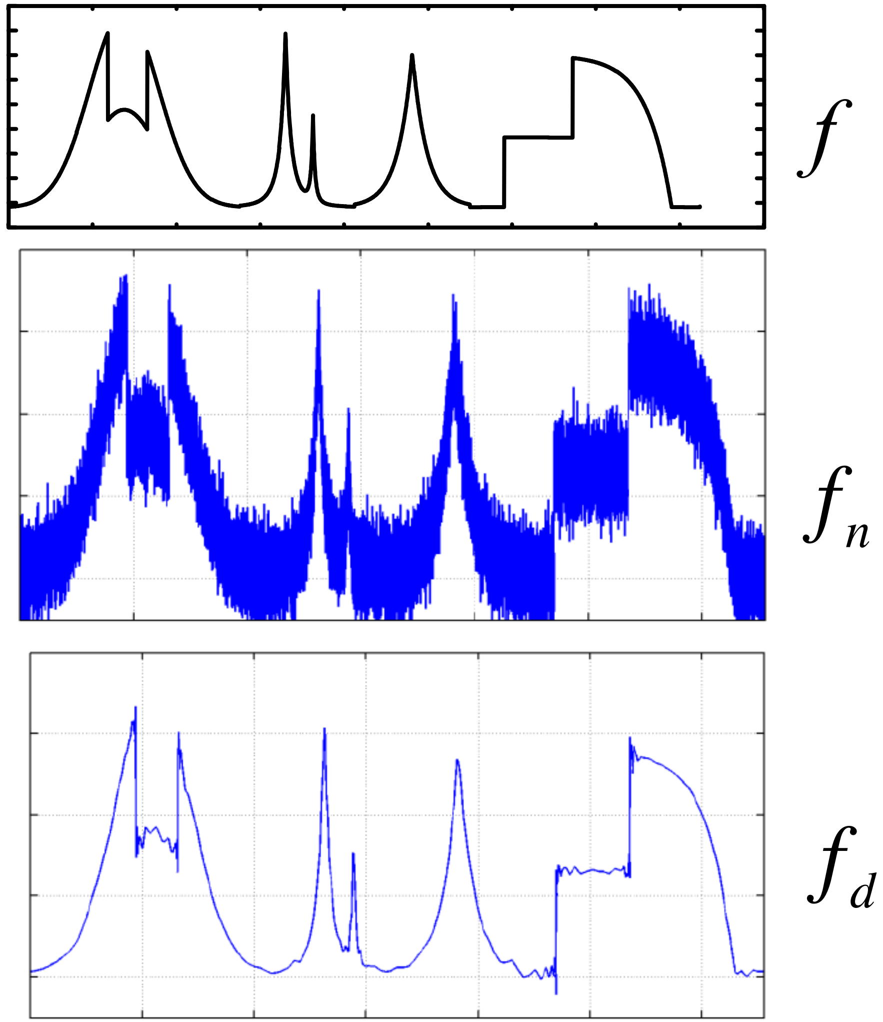
j scale,
i position

Estimation of the threshold :

$$\varepsilon_n = \sqrt{2 < f_n^{\ 2} > \ \ln(N)}$$

Wavelet reconstruction :

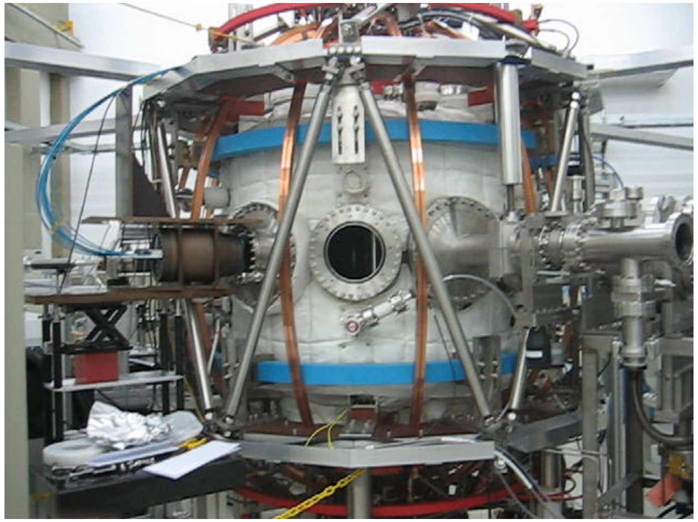
$$f_d = \sum_{ji : |\tilde{f}_{ji}| > \varepsilon_n} \tilde{f}_{ji} \psi_{ji}$$



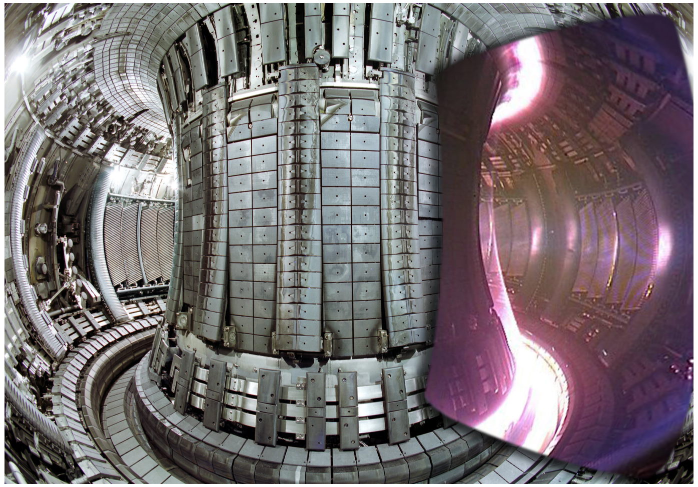
Donoho, Johnstone, Biometrika, ***81****, 1994*

Azzalini, M. F., Schneider, ACHA, ***18*** *(2), 2005*

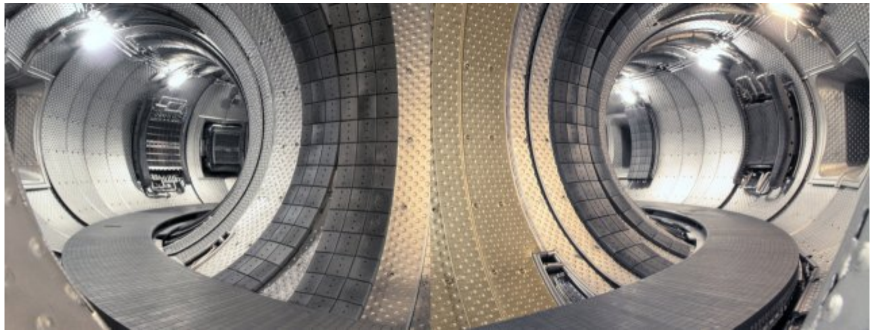
1. Application to plasma turbulence in tokamaks



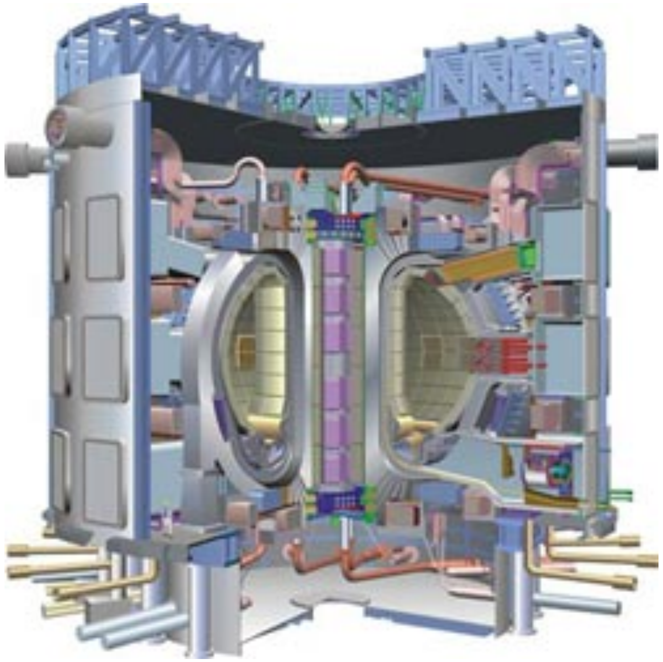
ETE, INPE (Brazil)



JET, Culham (Europe)



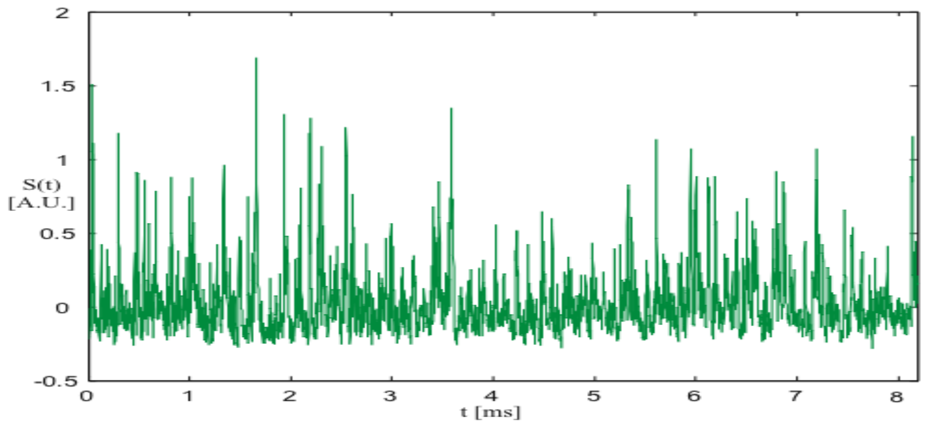
Tore-Supra, Cadarache (France)

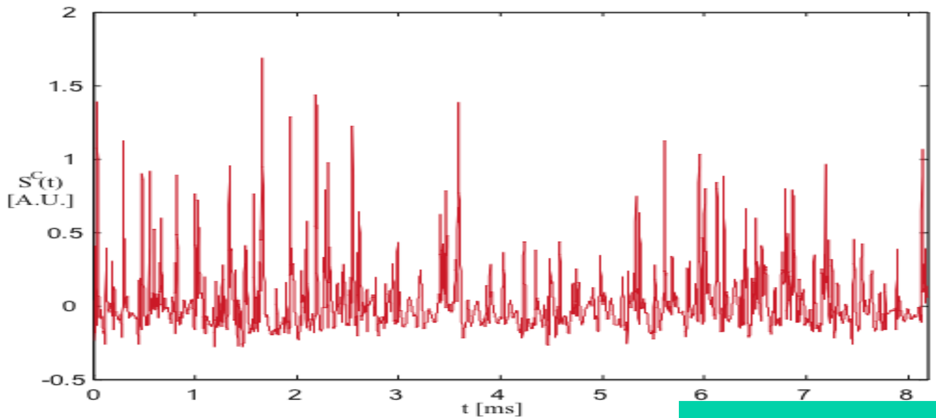


ITER (World)

Extraction of coherent structures SOL

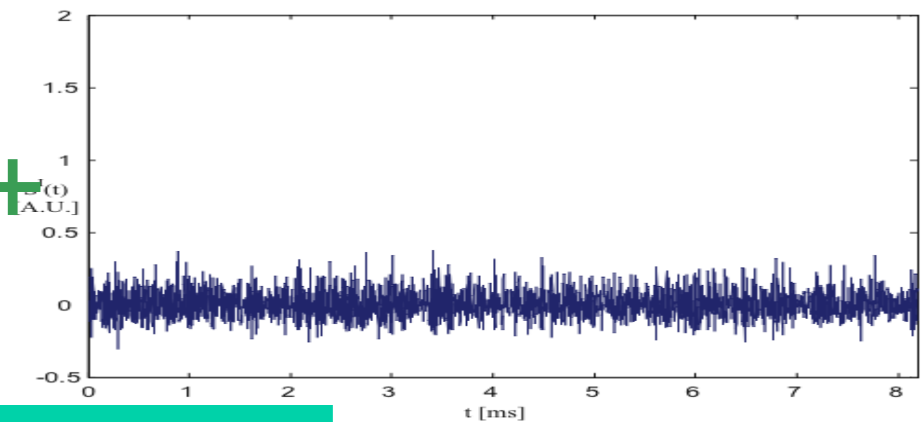
Ion density fluctuations measured by a fast reciprocating
Langmuir probe in the SOL of the tokamak Tore Supra
(Pascal Devynck, Tore-Supra, CEA-Cadarache)





Coherent

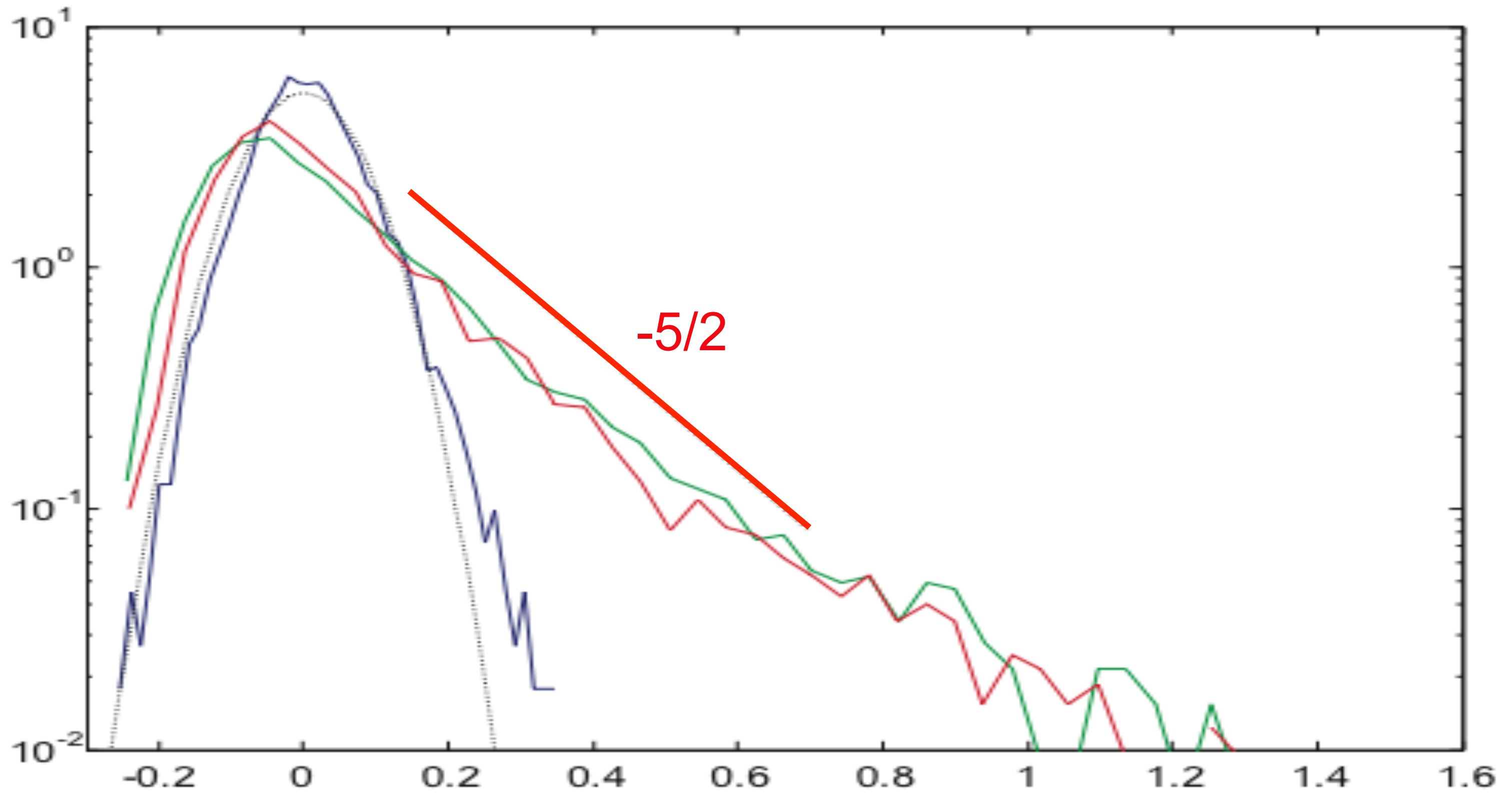
+



Incoherent

Farge, Schneider & Devynck
Phys. Plasmas, ***13****, 042304, 2006*

PDF of the density fluctuations

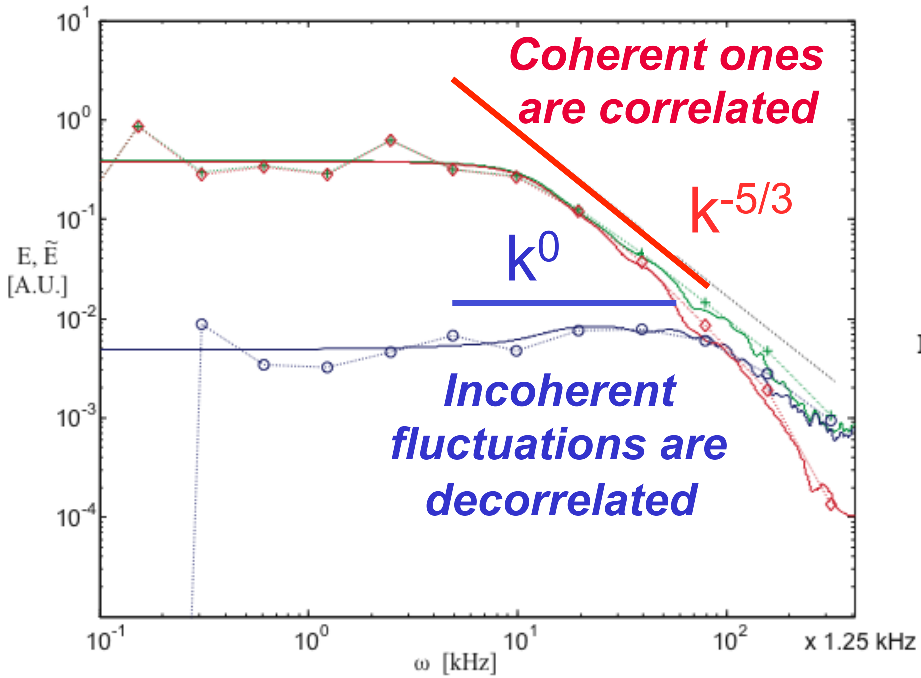


Total fluctuations = coherent + incoherent fluctuations

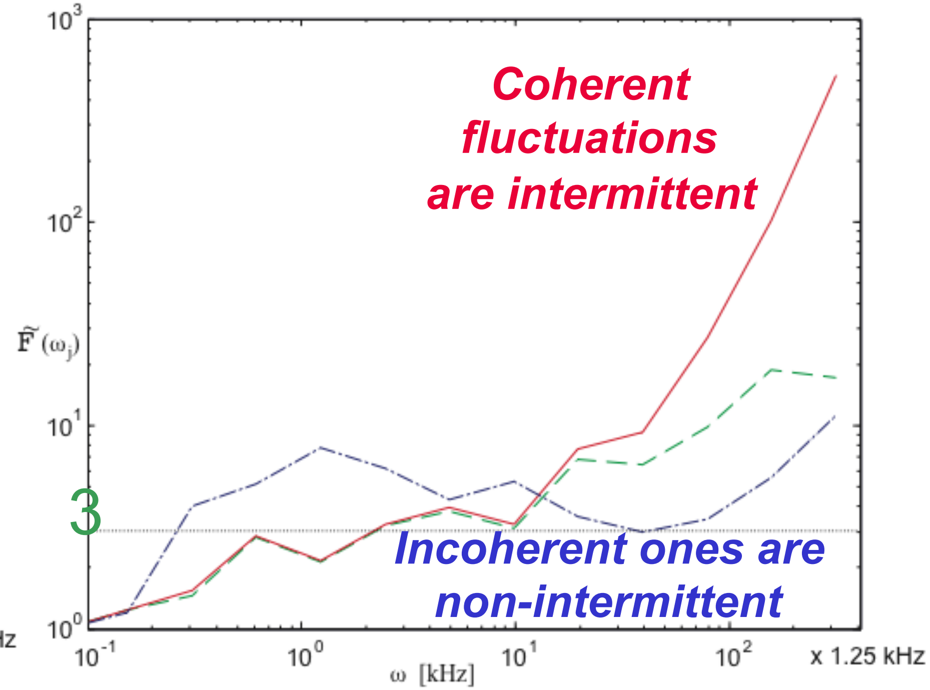
Farge, Schneider & Devynck, Phys. Plasmas, ***13****, 2006*

Correlation and intermittency

Scalogram
(stabilized periodogram)



Flatness versus scale
(from wavelet coefficients)



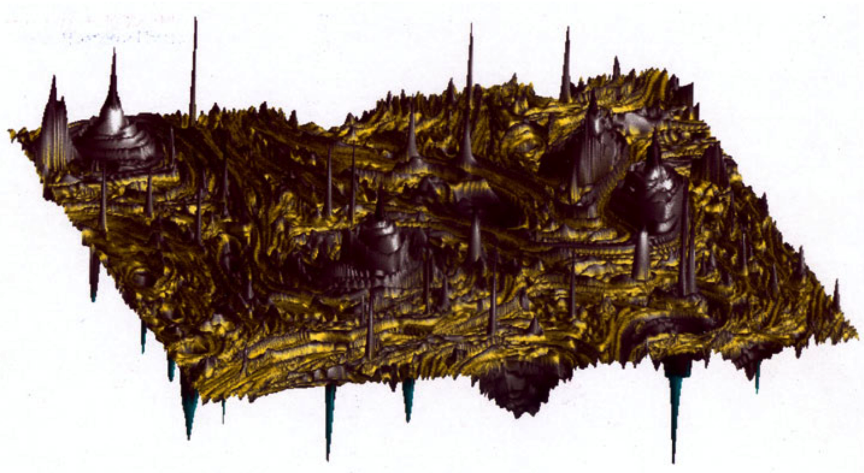
Total fluctuations = coherent + incoherent fluctuations

Farge, Schneider & Devynck, Phys. Plasmas, ***13****, 2006*

2. Applications in 2D fluid turbulent flows

DNS
$N=512^2$

0.2 % of coefficients
99.8 % of kinetic energy
93.6 % of enstrophy

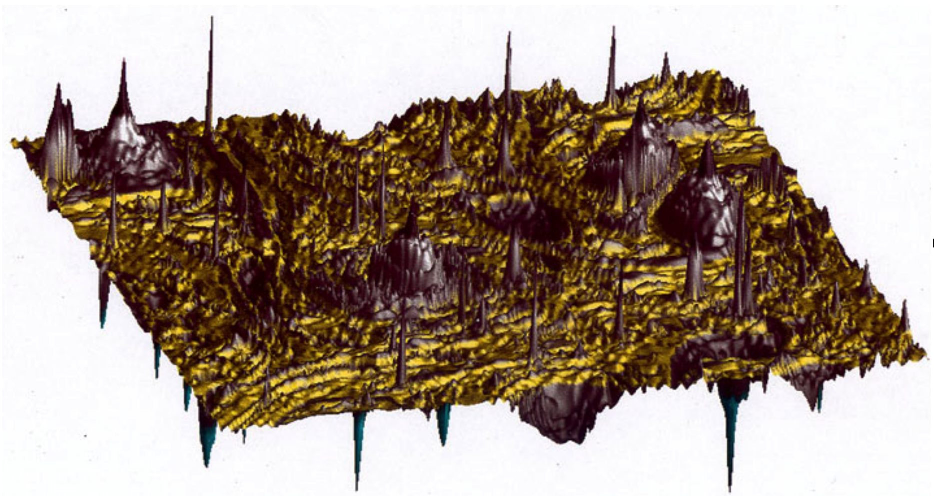


99.8 % of coefficients
0.2 % of kinetic energy
6.4 % of enstrophy

Coherent flow

Total flow

Incoherent flow



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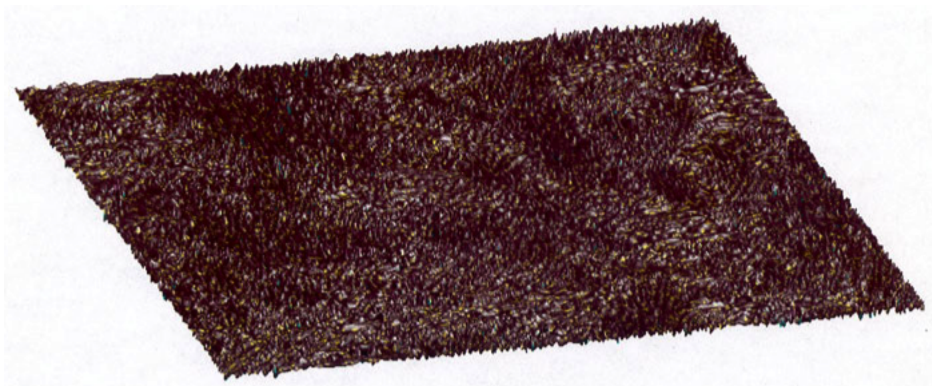
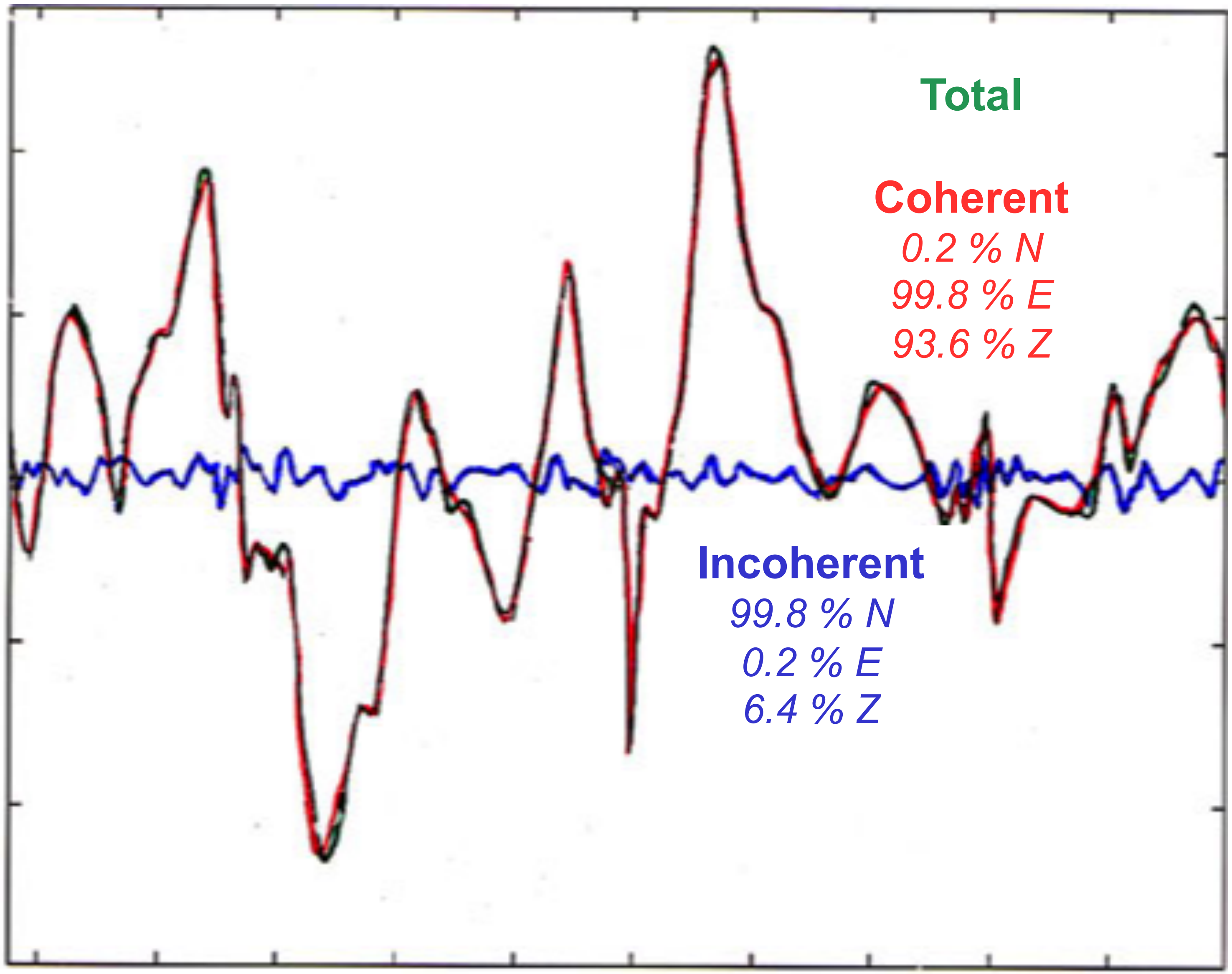


1D cut of the vorticity field

DNS
$N=512^2$

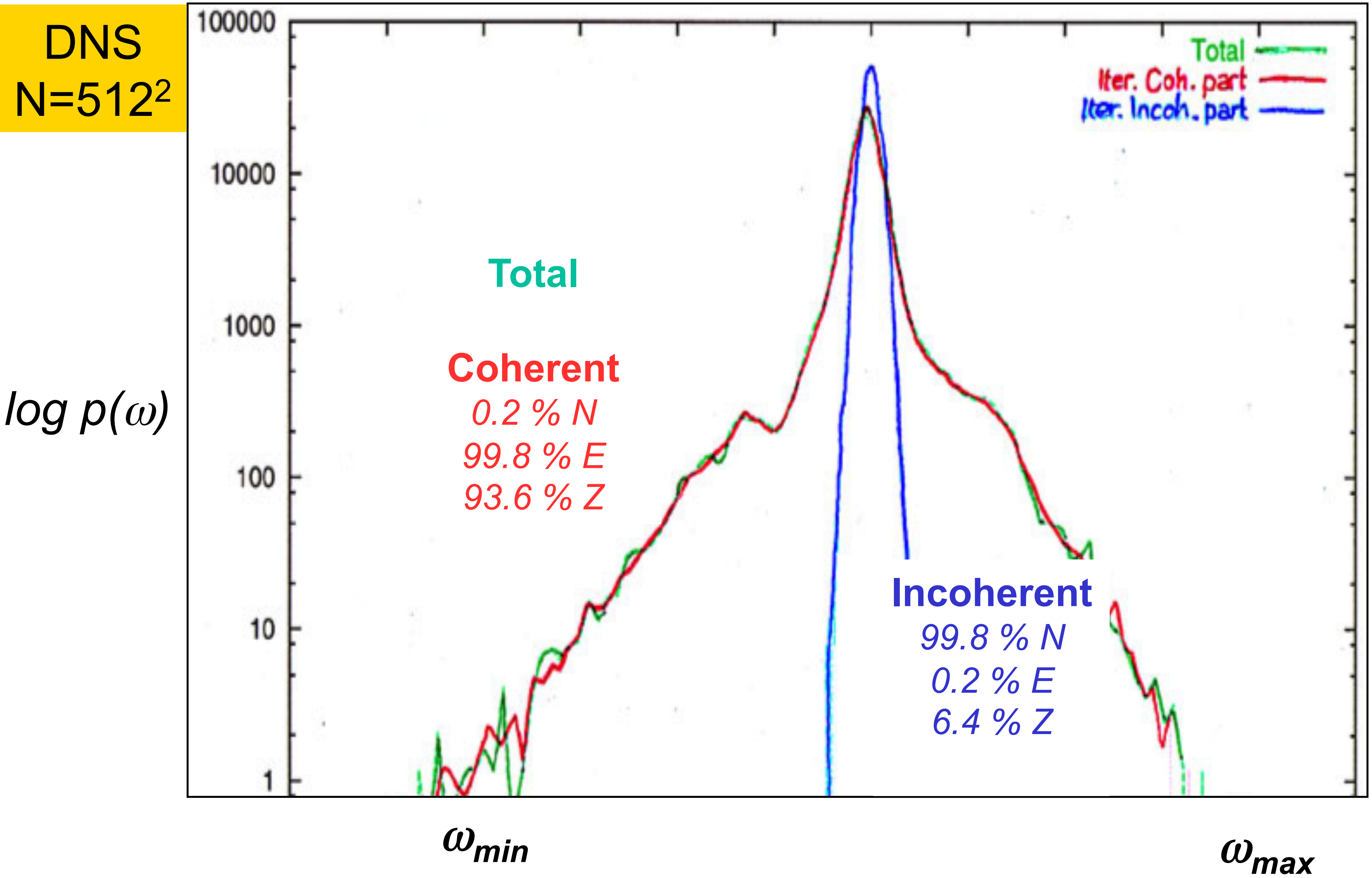
$\omega_t = \omega_c + \omega_i$

$Z_t = Z_c + Z_i$

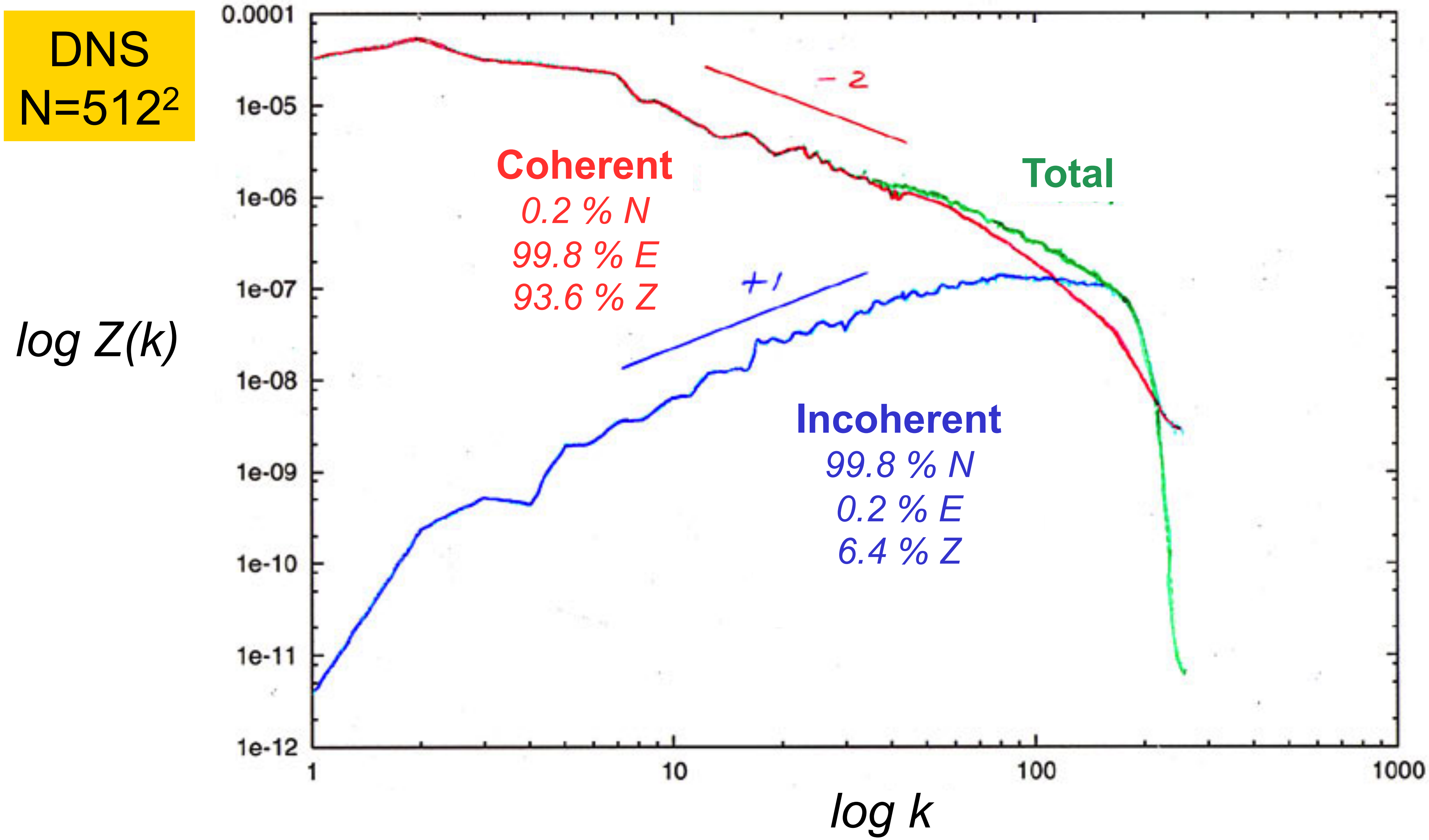


PDF of vorticity

DNS
$N=512^2$



Enstrophy spectrum



A posteriori proof of coherence

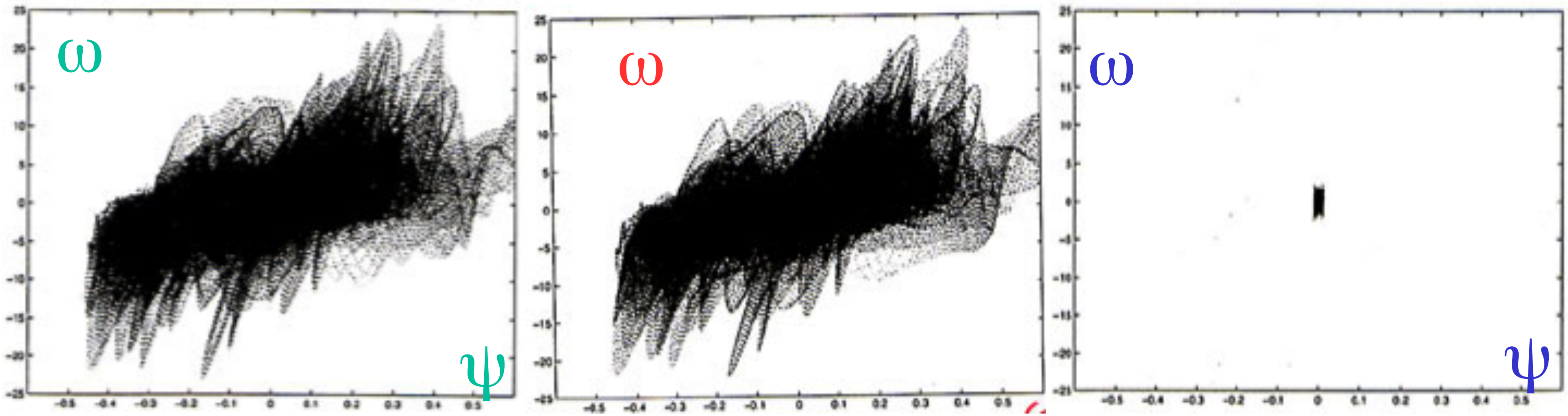
DNS
$N=512^2$

Coherent structures are
locally (in space and time)
steady solutions of Euler equation,
thus, for 2D flows :

$$\omega = \sinh(\psi)$$

Arnold, 1965,
Joyce & Montgomery, 1973
Robert & Sommeria, 1991

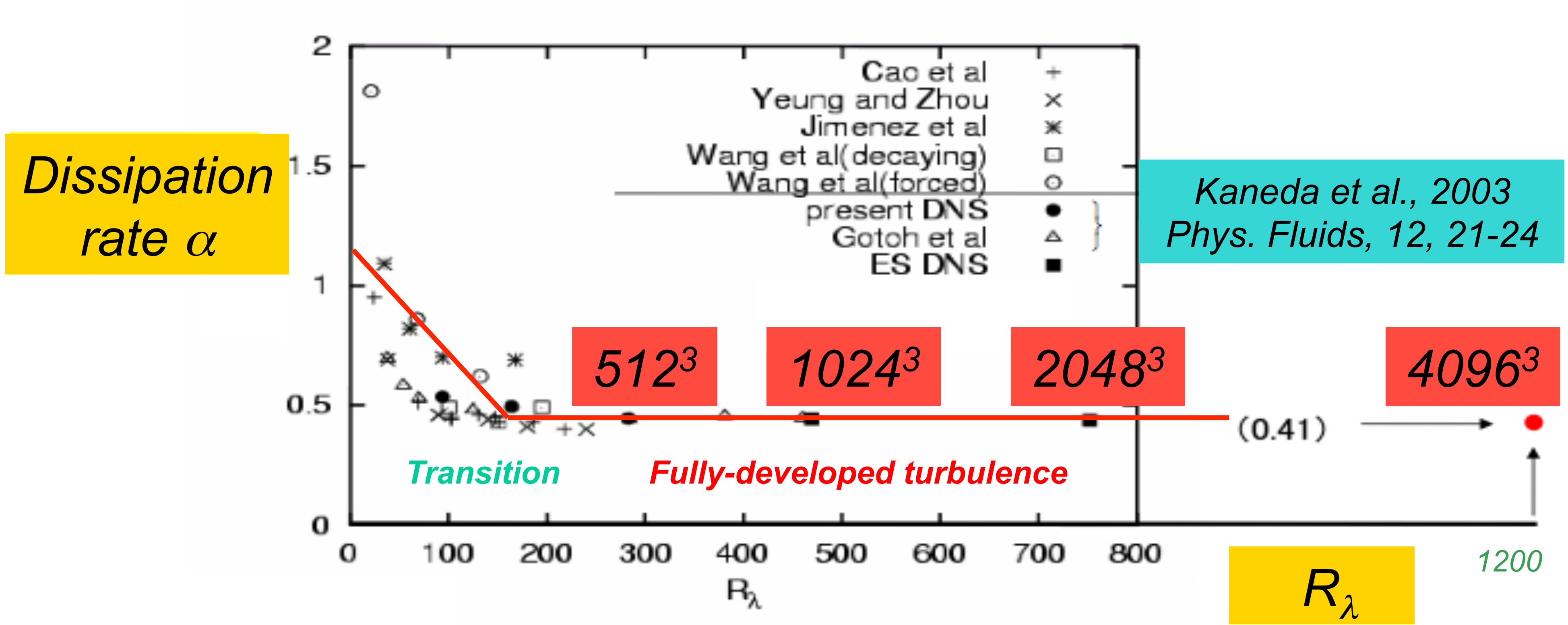
Total **Coherent** **Incoherent**



3. Application to 3D fluid turbulence

Normalized energy dissipation $\alpha \rightarrow$?

$\alpha = \epsilon L / u'^3$ as $\nu \rightarrow 0$, or Re $\rightarrow \infty$



Dissipation rate is independent of viscosity $\Rightarrow$ turbulent dissipation

How turbulent dissipation differs from viscous dissipation?

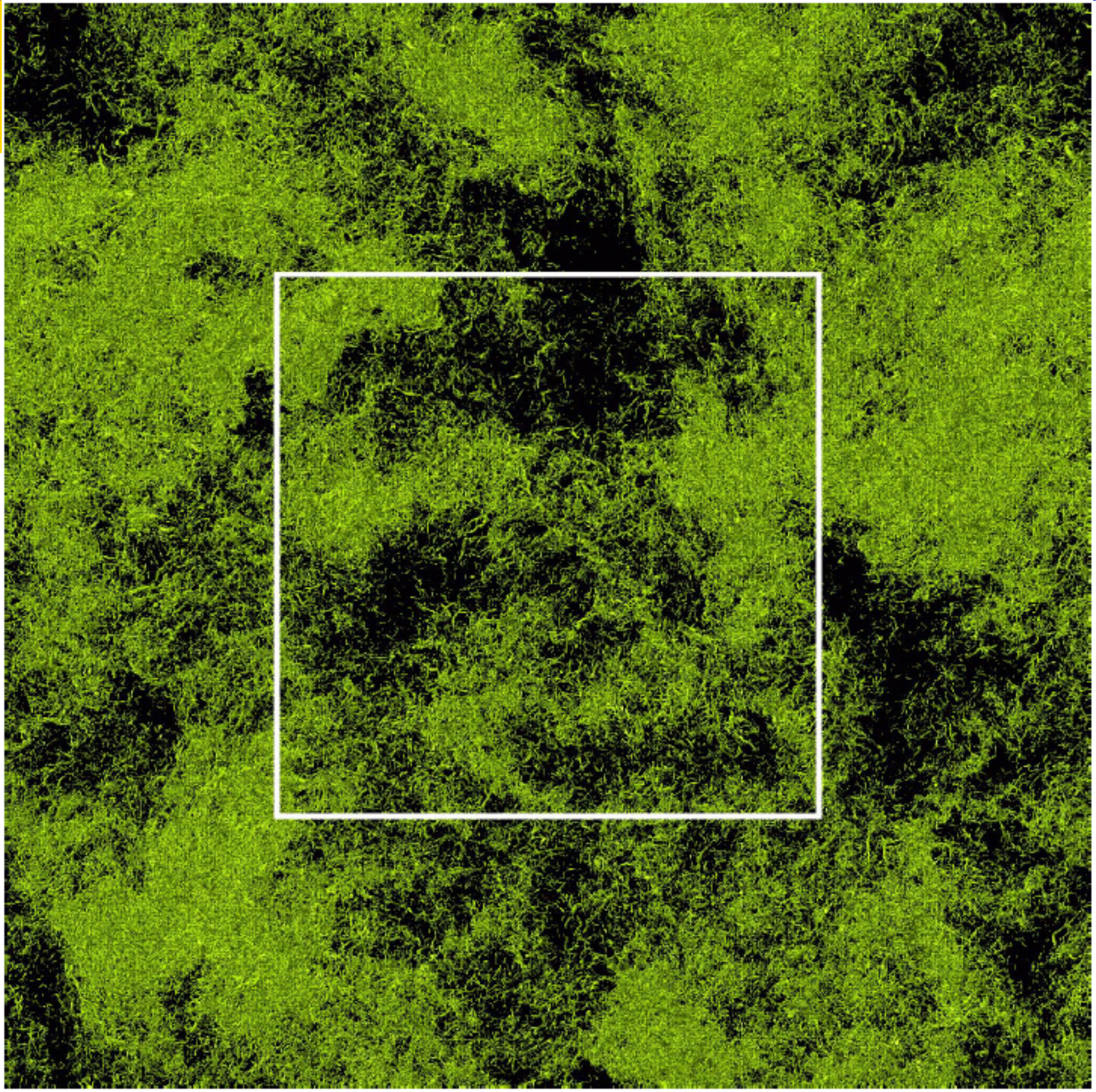
3D homogeneous isotropic turbulence

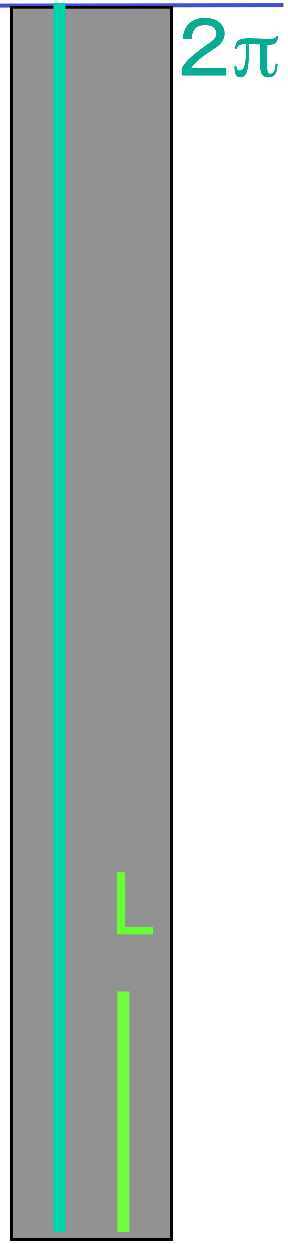
Resolution
$N=2048^3$

L,
integral
scale

*Computed
in 2002
on ES1
14 Tflops
10 Tbytes*

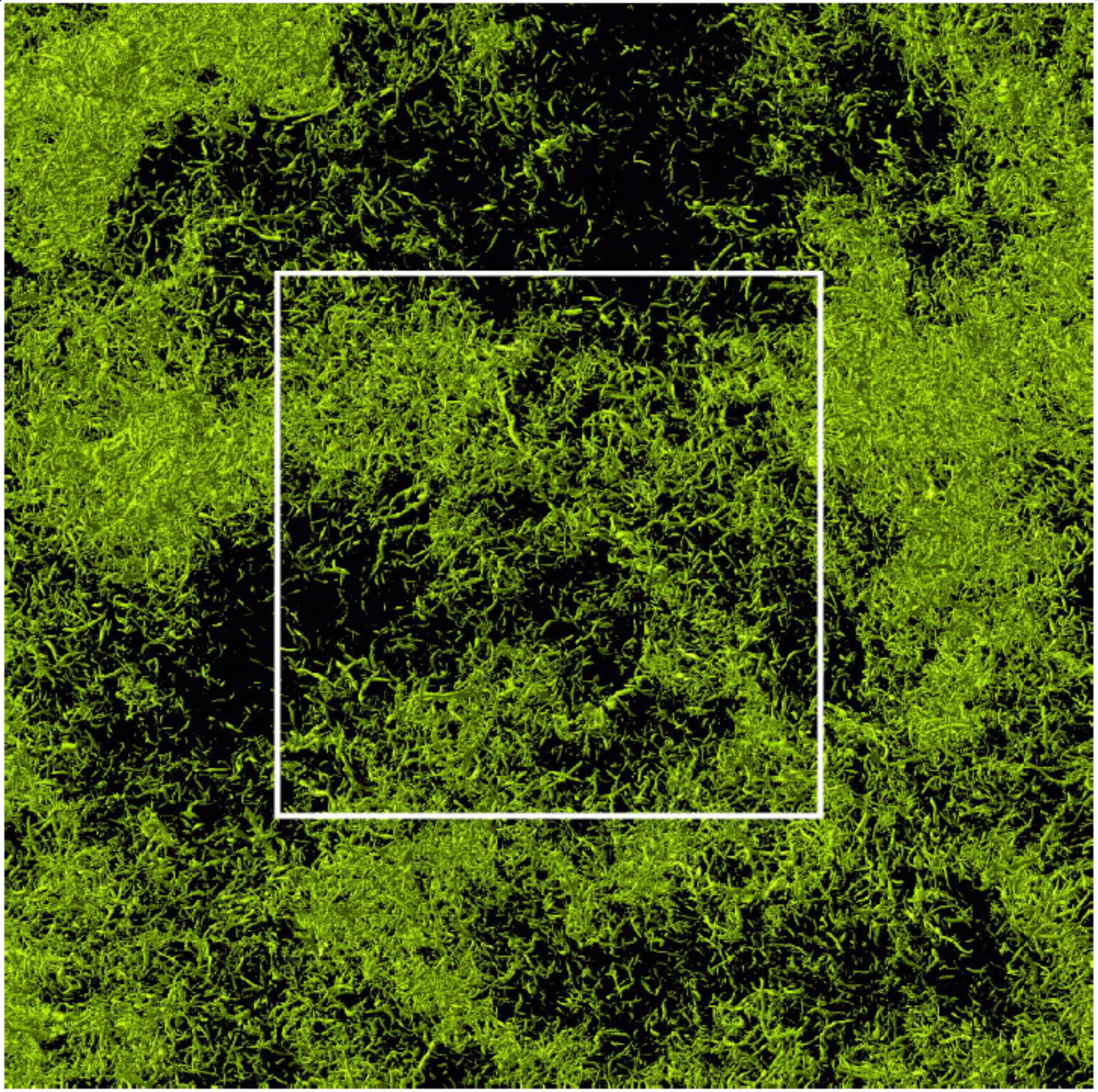
*Kaneda,
Ishihara
et al., 2003,
Phys. Fluids,
12, 21-24*





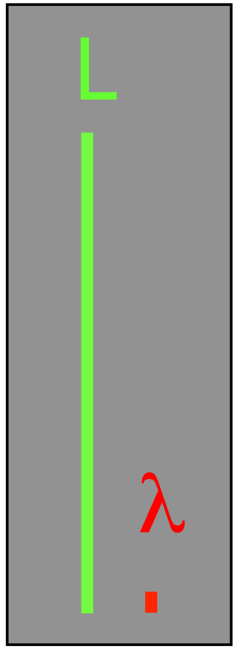
Zoom (sub-cube 1024^3)

Resolution
$N=2048^3$



Kaneda, Ishihara et al., 2003, Phys. Fluids, ***12****, 21-24*

L,
Integral
scale



λ,
Taylor
micro-
scale

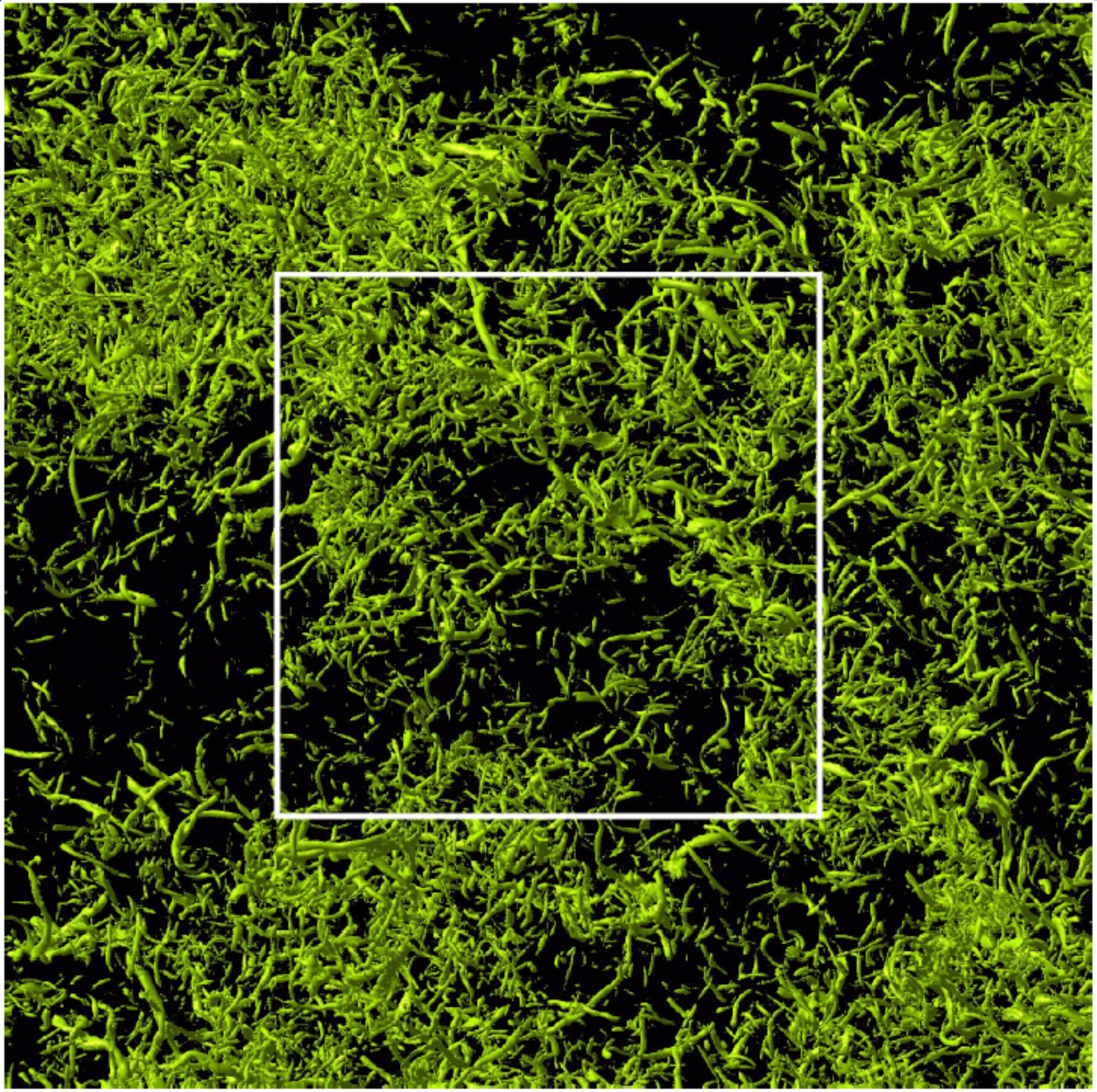
Zoom (sub-cube 512^3)

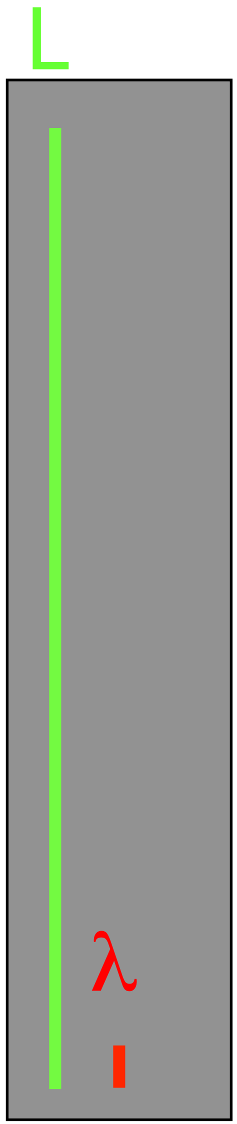
Resolution
N=2048^3

L,
integral
scale

λ,
Taylor
macro-
scale

*Kaneda,
Ishihara
et al., 2003,
Phys. Fluids,*
***12**, 21-24*





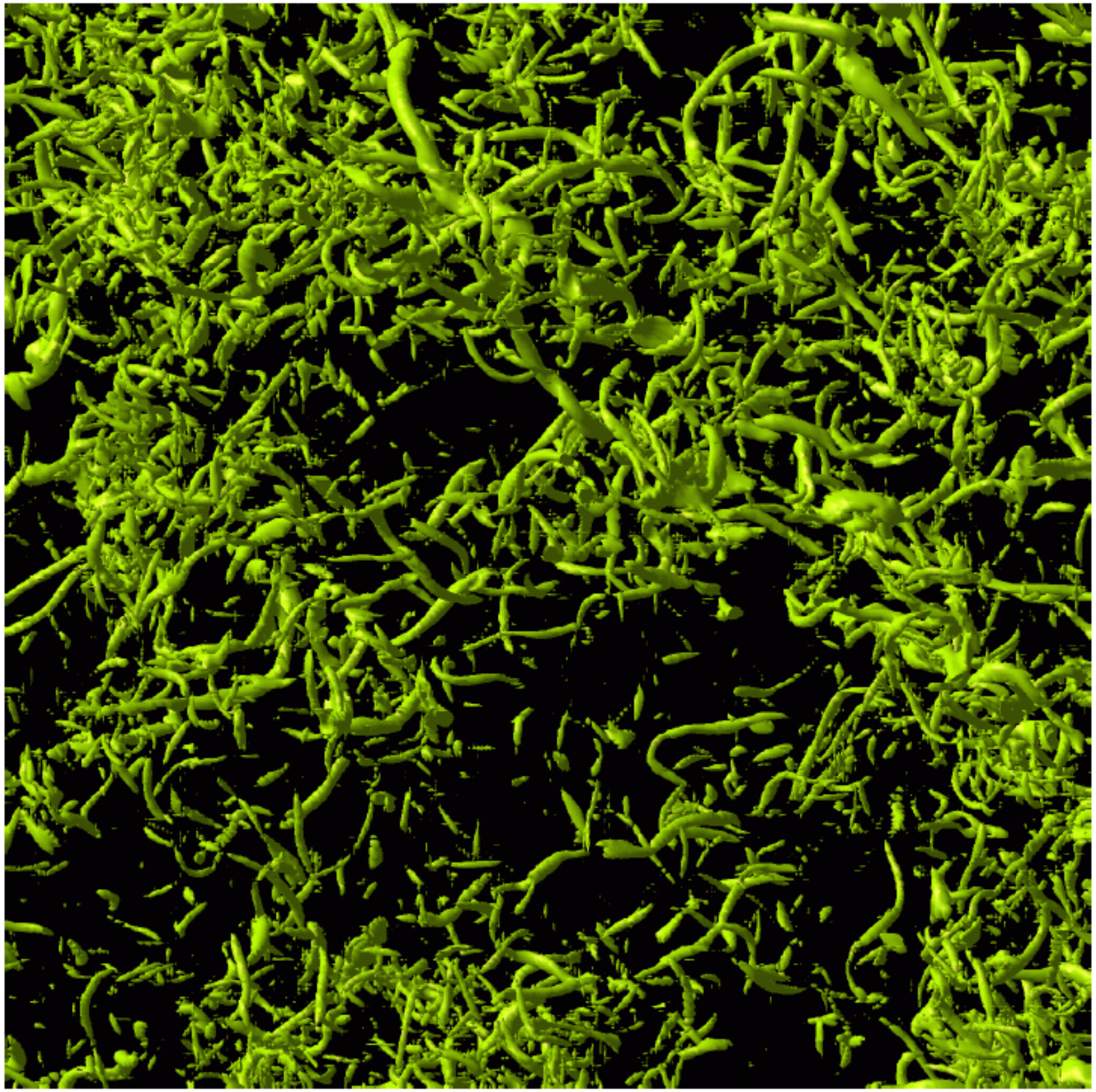
Zoom (sub-cube 256^3)

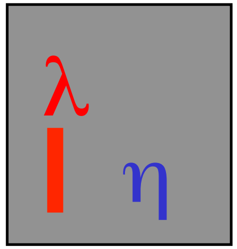
Resolution
$N=2048^3$

λ,
Taylor
macro-
scale

η,
Kolmogorov
dissipative
scale

Kaneda,
Ishihara
et al., 2003,
Phys. Fluids,
***12**, 21-24*





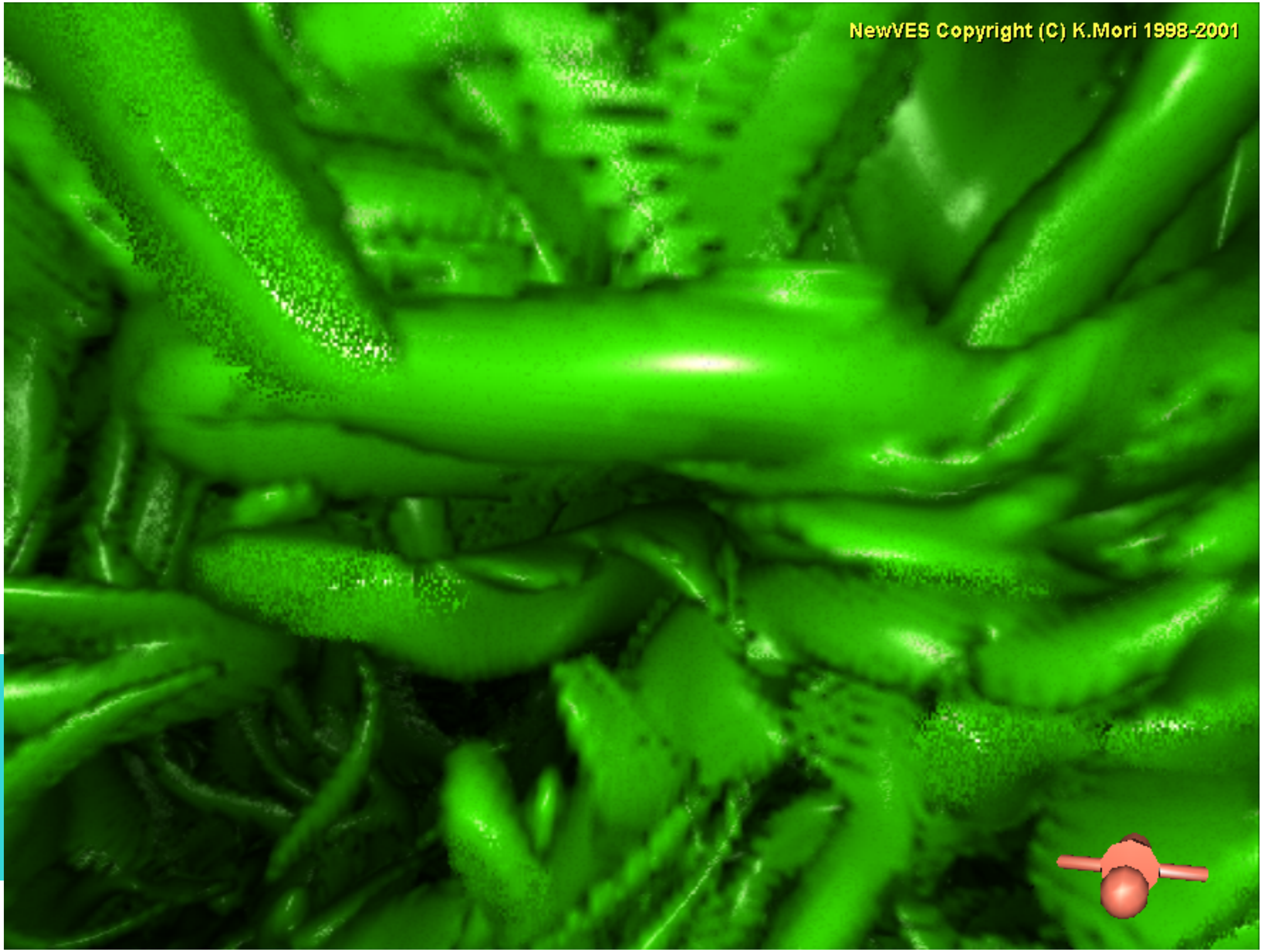
Zoom (sub-cube 128^3)

DNS
$N=2048^3$

Kaneda, Ishihara et al., 2003, Phys. Fluids, ***12****, 21-24*

Zoom (sub-cube 64^3)

DNS
$N=2048^3$

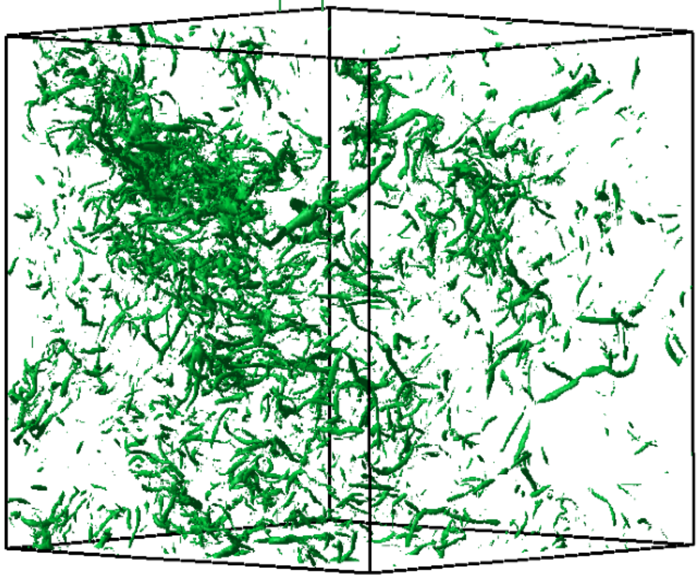


Kaneda, Ishihara et al., 2003, Phys. Fluids, ***12****, 21-24*

Extraction of coherent structures in 3D flows

DNS
$N=2048^3$

$|\omega|=5\sigma$



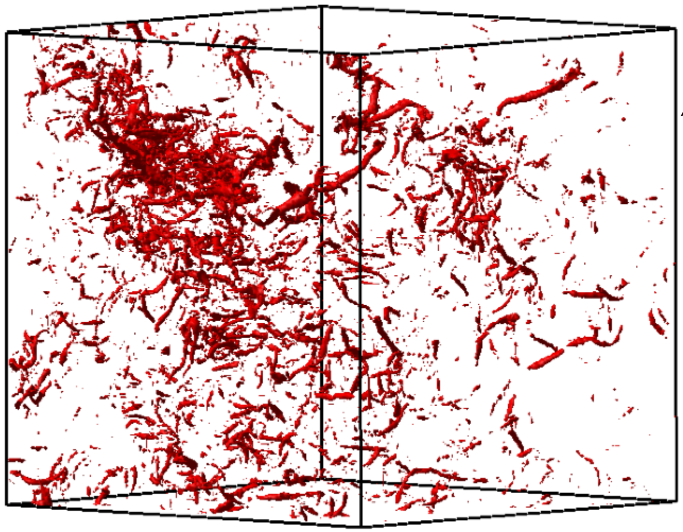
with $\sigma=(2Z)^{1/2}$

Coherent vorticity

2.6 % N coefficients
80% enstrophy
99% energy

Incoherent vorticity

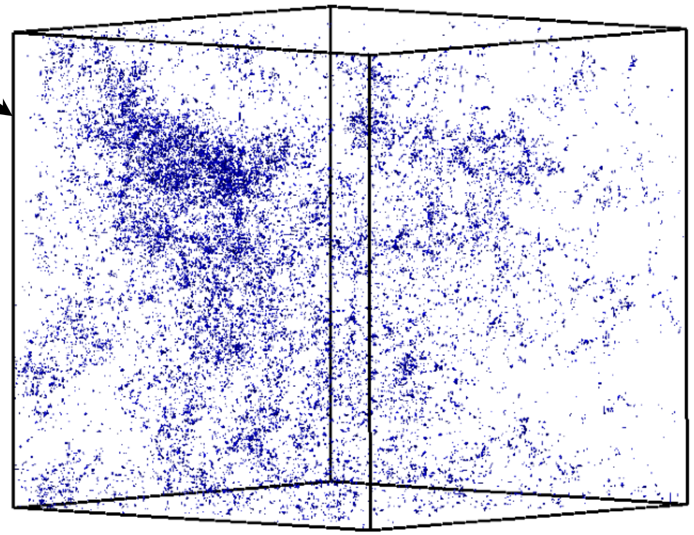
97.4 % N coefficients
20 % enstrophy
1% energy



$|\omega|=5\sigma$

Total vorticity

+

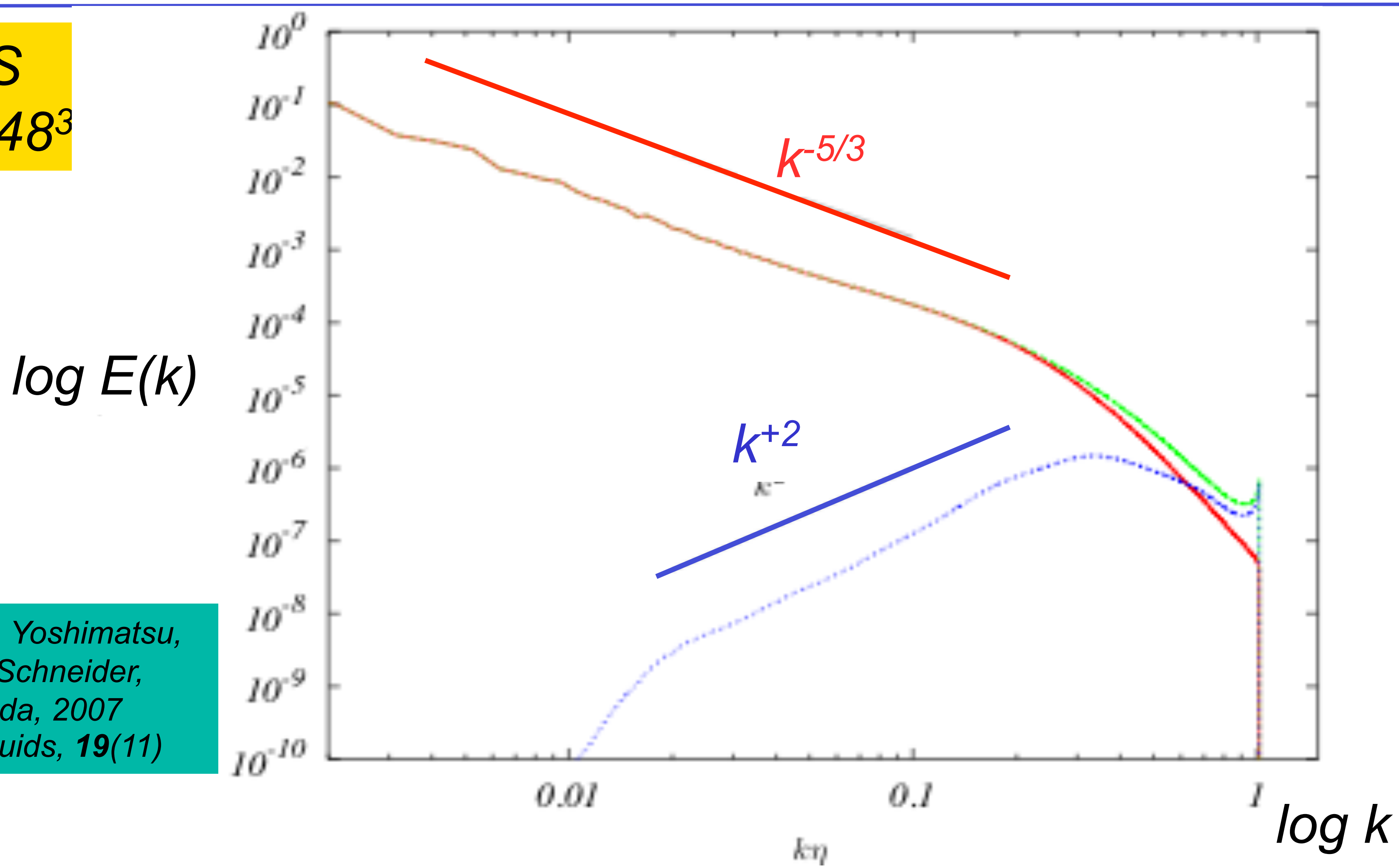


$|\omega|=5/3\sigma$

*Okamoto, Yoshimatsu, Schneider, Farge, Kaneda, 2007, Phys. Fluids, **19**, 1159*

Energy spectrum

DNS
N=2048[3]



Okamoto , Yoshimatsu, Farge, Schneider, Kaneda, 2007
Phys. Fluids, **19**(11)

2.6 % N coefficients
80% enstrophy
99% energy

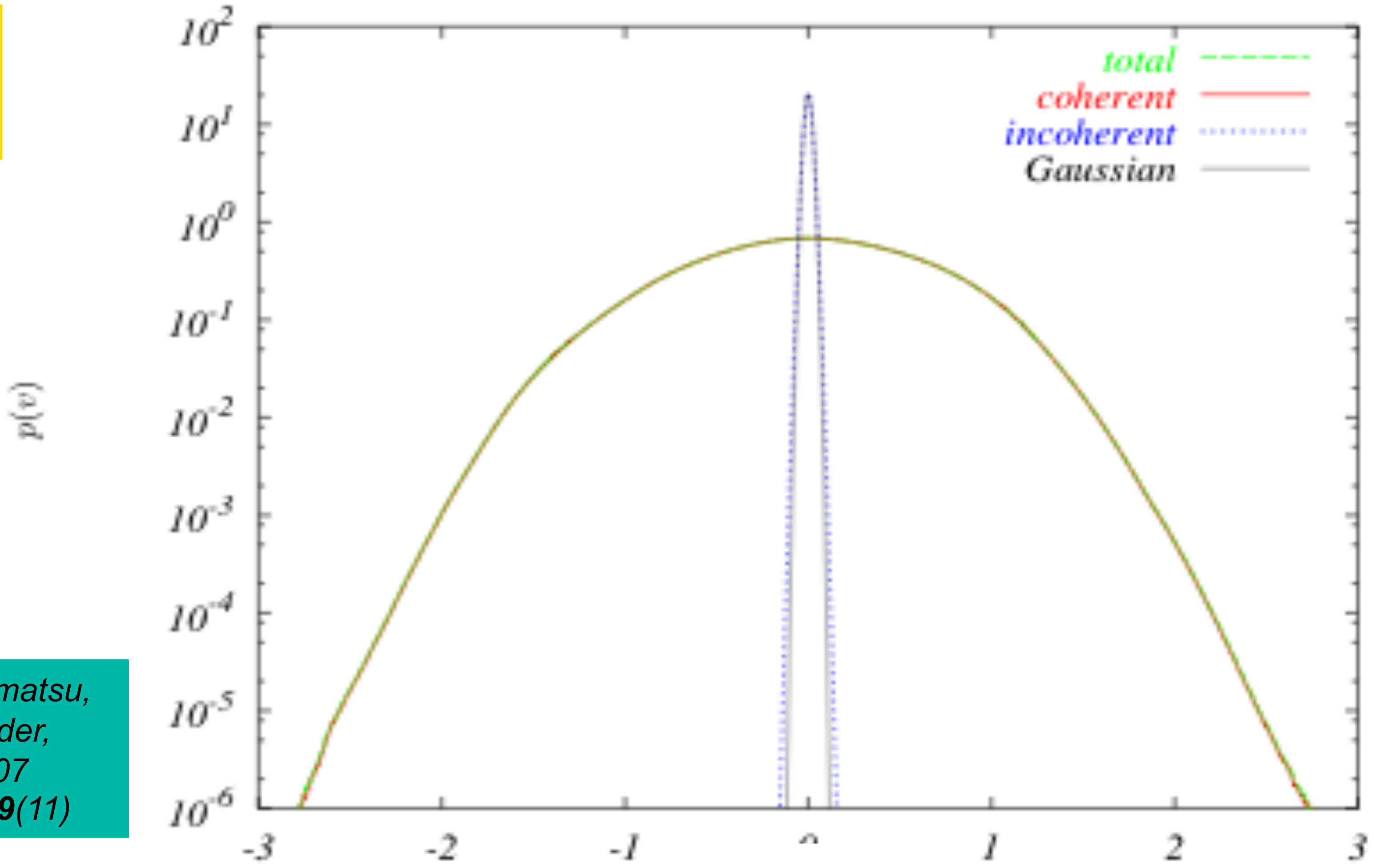
Multiscale Coherent
$k^{-5/3}$ scaling, i.e.
long-range correlation

Multiscale Incoherent
k^{+2} scaling, i.e.
energy equipartition

PDF of velocity

DNS
N=2048³

log p(v)



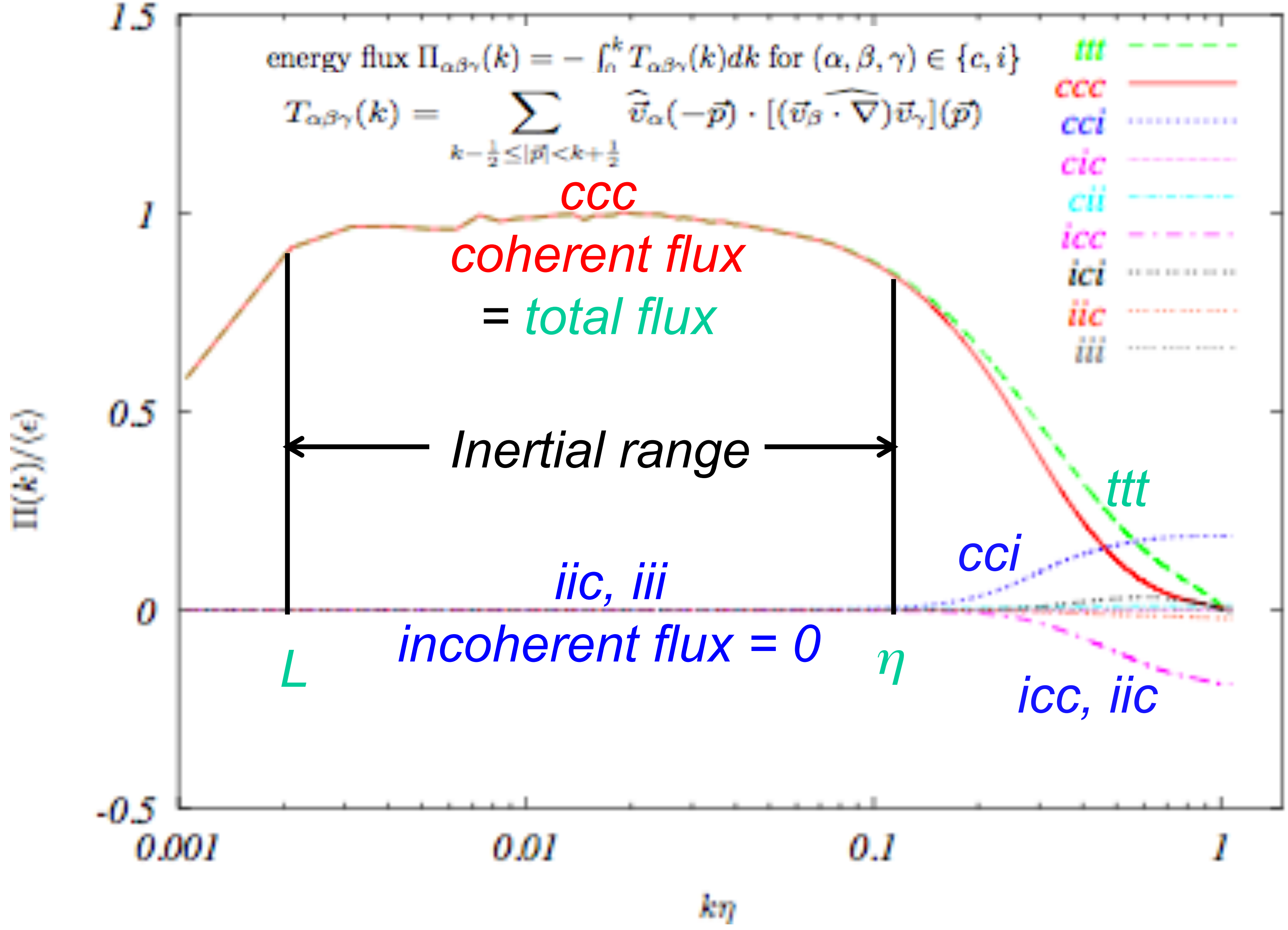
V

*Okamoto , Yoshimatsu, Farge, Schneider, Kaneda, 2007 Phys. Fluids, **19**(11)*

2.6 % N coefficients
99% energy

The total and coherent flows have the same extrema.
The incoherent flow has a Gaussian PDF, therefore its effect should be easy to model

Nonlinear transfers and energy fluxes



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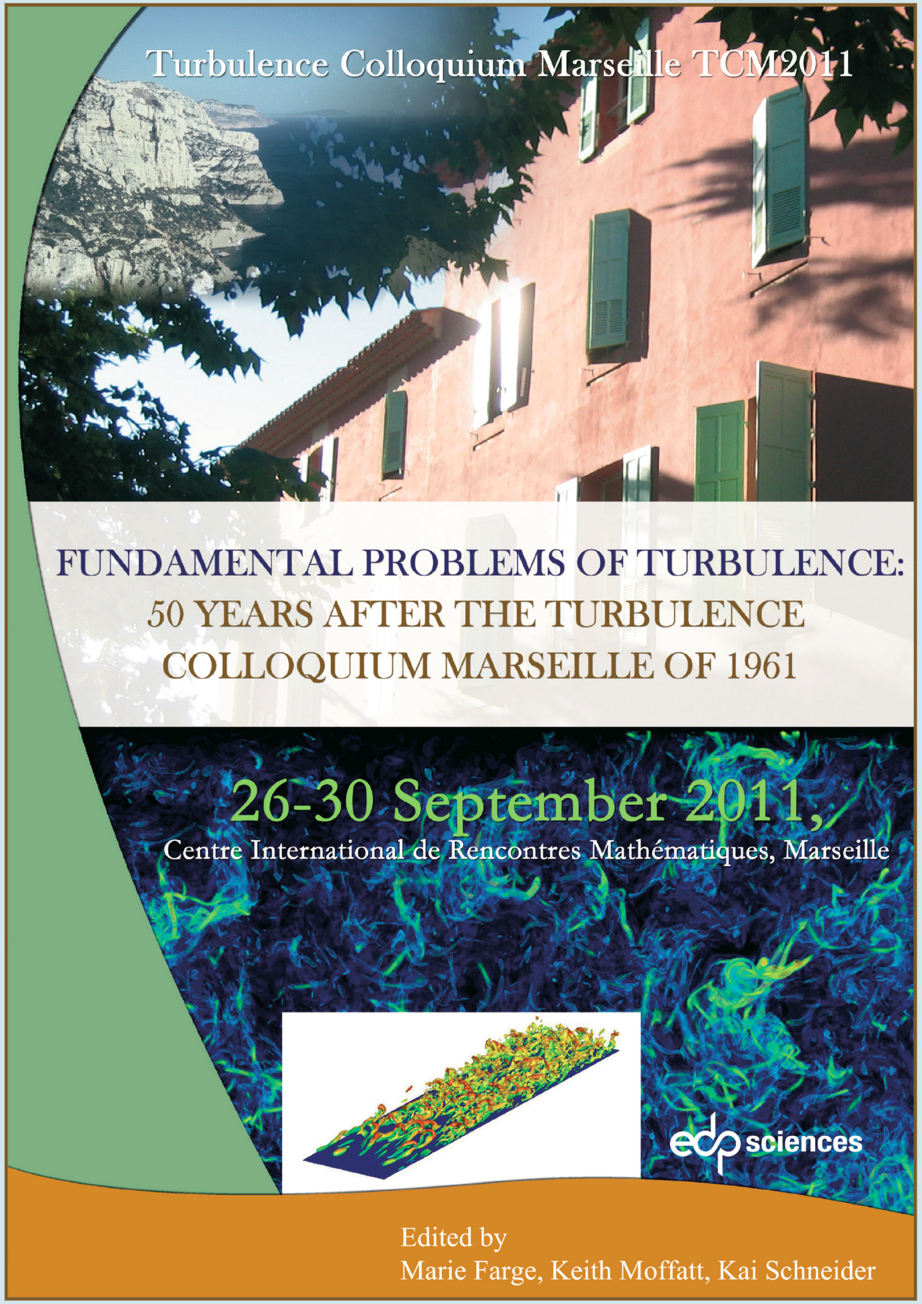

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Marie Farge is CNRS Research Director at École Normale Supérieure, Paris, France.

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