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On the inverse cascade of magnetic helicity at sub-ion scale

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Present Achievements and New Frontiers in Space Plasmas

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Entanglement of helicity and energy in kinetic Alfvén wave/whistler turbulence

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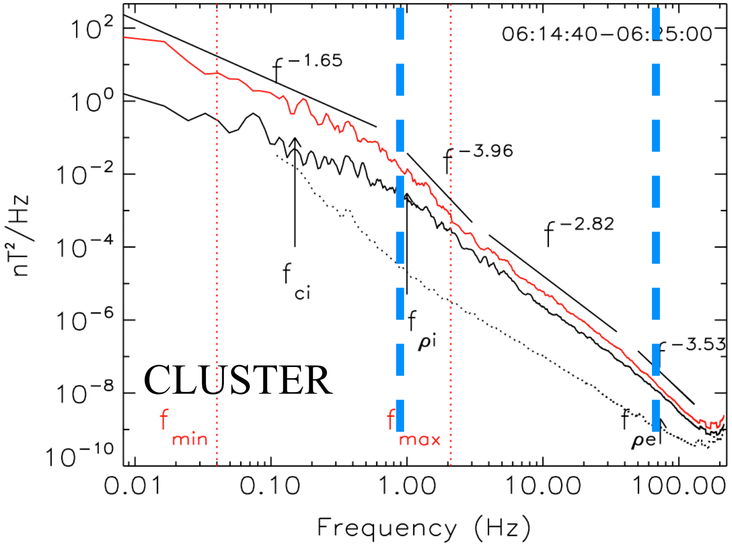
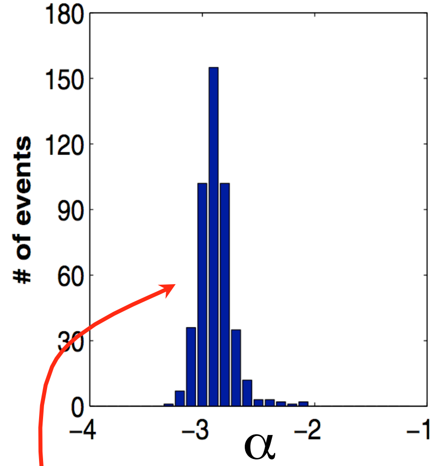
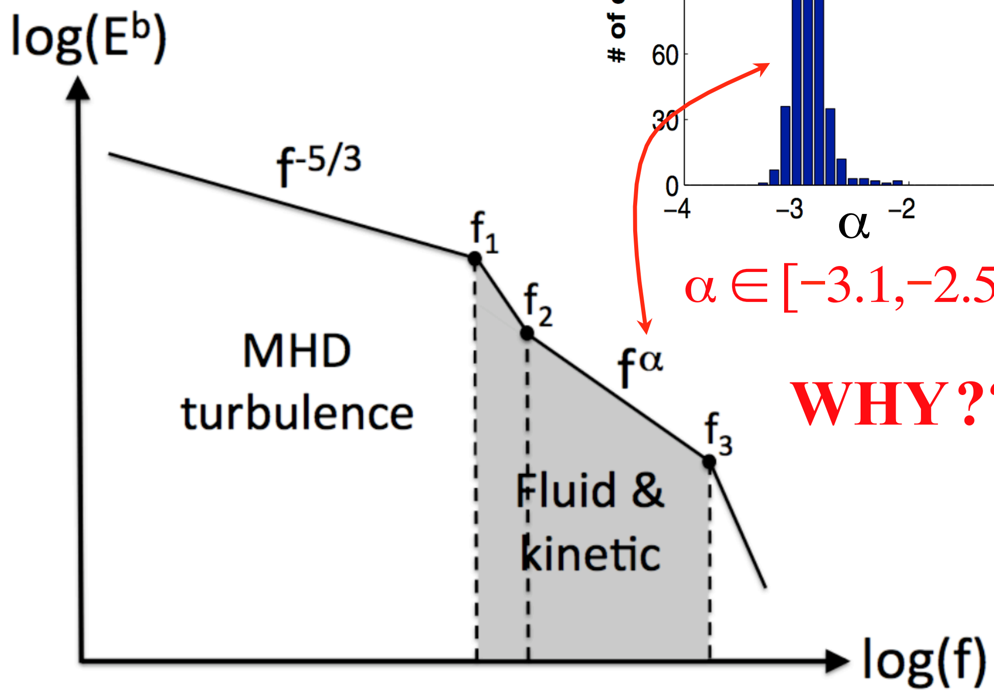
Motivation: solar wind turbulence

[eg. Alexandrova et al., PRL, 2009; Sahraoui et al., PRL, 2010]



[Smith et al., ApJ, 2006]
[Sahraoui et al., ApJ, 2013]





$\alpha \in [-3.1, -2.5]$

WHY??

Signature of reduced magnetic helicity at sub-ion scales
[eg. Howes et al., ApJL, 2010]

What is the origin of the scaling laws of the magnetic fluctuation spectra observed beyond f_2 ?

Assumption of the talk

$$\boxed{\text{Power laws are due to turbulence}}$$

Kinetic effects can be neglected
and
fluid models can be used

Generalized Ohm's law

Ideal MHD

$$\overbrace{\mathbf{E} + \mathbf{V}\times\mathbf{B}} - \frac{\mathbf{J}\times\mathbf{B}}{ne} = \mu_0 \eta \mathbf{J}$$

Hall effect

→ It leads to the Hall MHD equations (dispersive effects)

$$\left|\frac{\nabla\times\left(\frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{\mu_0 ne}\right)}{\nabla\times(\mathbf{u}\times\mathbf{B})}\right| \sim \left|\frac{\mathbf{B}/\ell}{\mu_0 ne\,\mathbf{u}}\right| \sim \frac{B}{\mu_0 ne\,\ell\,u} \sim \frac{d_i}{\ell}\,\frac{b}{u}$$

Oblique whistler / Kinetic Alfvén Wave

$$b_\perp \sim b_\| \ll b_0 \qquad k_\perp \gg k_\|$$

[Schekochihin et al., ApJSS, 2009]

	Hall MHD	EMHD	Reduced EMHD
EQUATIONS	$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{u}) = 0\,,$ $\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla P + \frac{1}{\mu_0}(\nabla\times\mathbf{b})\times\mathbf{b}\,,$ $\frac{\partial \mathbf{b}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{b}) - \nabla\times\left(\frac{(\nabla\times\mathbf{b})\times\mathbf{b}}{\mu_0 n e}\right)\,,$	$\beta \gg 1$ $\partial_t \mathbf{b} + d_i b_0 \partial_\| (\nabla_\perp \times \mathbf{b}) = -d_i \nabla_\perp \times [\mathbf{b}_\perp \cdot \nabla_\perp \mathbf{b}]$	$\mathbf{u}_i = 0 \quad \nabla\cdot\mathbf{u}_i \neq 0 \quad \lambda = \frac{\beta_i(1+Z/\tau)}{2+\beta_i(1+Z/\tau)} \quad \tau = T_i/T_e \quad Z = n_e/n_i$ $\partial_t \mathbf{b}_\perp + \frac{c}{4\pi e n_{0e}} b_0 \partial_\| (\nabla_\perp \times \mathbf{b}_\|) = -\frac{c}{4\pi e n_{0e}} \nabla_\perp \times [\mathbf{b}_\perp \cdot \nabla_\perp \mathbf{b}_\|]\,,$ $\partial_t \mathbf{b}_\| + \lambda \frac{c}{4\pi e n_{0e}} b_0 \partial_\| (\nabla_\perp \times \mathbf{b}_\perp) = -\lambda \frac{c}{4\pi e n_{0e}} \nabla_\perp \times [\mathbf{b}_\perp \cdot \nabla_\perp \mathbf{b}_\perp]\,,$
WAVES	Fast, slow (L), oblique whistler (R) KAW (R) : $\omega/\omega_{ci} \ll 1$ $\omega = \sqrt{\frac{\beta}{1+\beta}} d_i k_\perp b_0 k_\|$	Oblique whistler (R) : $\omega = d_i k_\perp b_0 k_\|$	$d_i = \rho_i/\sqrt{\beta_i}$ KAW (R) : $\omega = \sqrt{\frac{\beta_i(1+Z/\tau)}{2+\beta_i(1+Z/\tau)}} d_i k_\perp b_0 k_\|$

$$\mathbf{b}_\perp \to \frac{\tilde{\mathbf{b}}_\perp}{\sqrt{\lambda}}\,, \qquad \mathbf{b}_0 \to \frac{\tilde{\mathbf{b}}_0}{\sqrt{\lambda}}$$

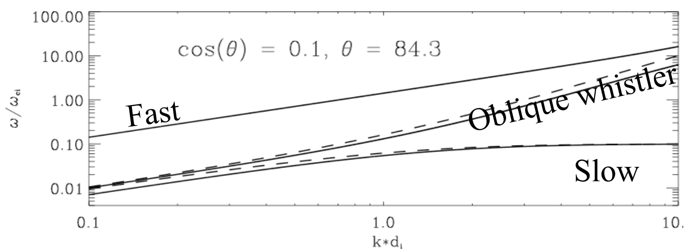
$$b_\perp \sim b_\| \ll b_0$$

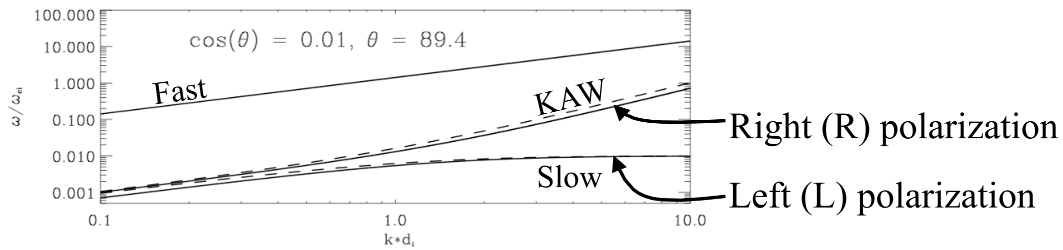
[SG & Meyrand, JPP, 2015]

Same dynamics for oblique whistler and KAW !

$$\boxed{\partial_t \tilde{\mathbf{b}} + d_i \tilde{b}_0 \partial_\| (\nabla_\perp \times \tilde{\mathbf{b}}) = -d_i \nabla_\perp \times \left[\tilde{\mathbf{b}}_\perp \cdot \nabla_\perp \tilde{\mathbf{b}}\right]}$$

Hall MHD:

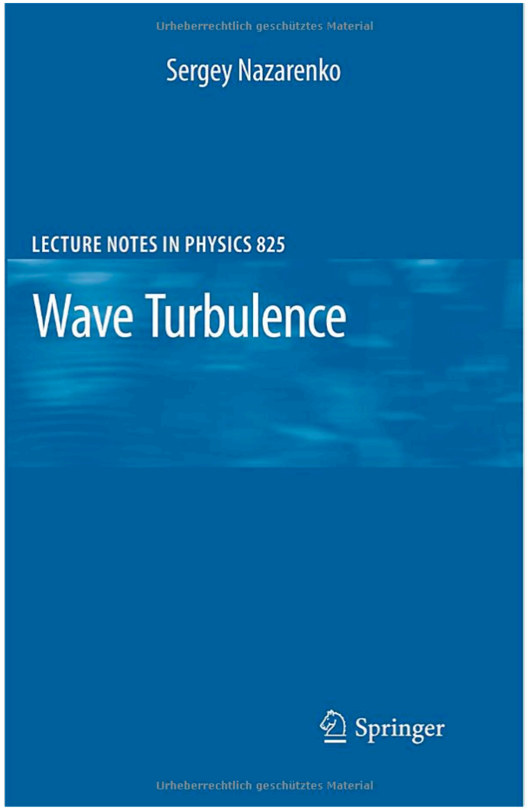


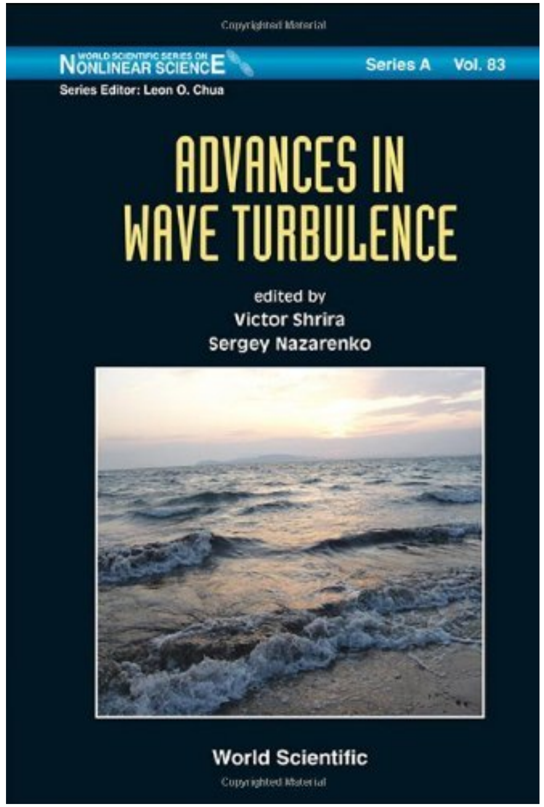


All the warm discussion about oblique whistler vs KAW does not matter for the dynamics !

Weak wave turbulence

- ✓ Statistical theory of weakly nonlinear dispersive waves
- ✓ Exact solutions can be found *via* the Zakharov transform





Rigorous mathematical derivation

Weak turbulence = theory for weakly nonlinear dispersive waves

$$\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t} = \mathcal{L}(\mathbf{u}) + \varepsilon \mathcal{N}(\mathbf{u},\mathbf{u}) \qquad \varepsilon \ll 1$$

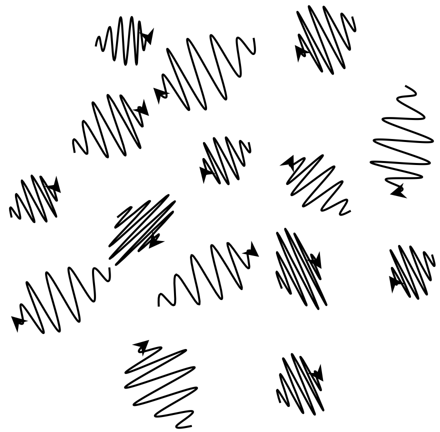
$$\mathbf{A}(\mathbf{k},t) = \frac{1}{(2\pi)^3}\int_{\mathbf{R}^3} \mathbf{u}(\mathbf{x},t)\exp(-i\mathbf{k}\cdot\mathbf{x})d\mathbf{x}$$

$$\mathbf{A}(\mathbf{k},t) = \mathbf{a}(\mathbf{k},t)e^{-i\omega_k t}$$

$$\frac{\partial a_j(\mathbf{k})}{\partial t} = \varepsilon \int_{\mathbf{R}^6} \mathcal{H}_{jmn}^{\mathbf{kpq}} a_m(\mathbf{p}) a_n(\mathbf{q}) e^{i\Omega_{k,pq}t}\delta_{k,pq} d\mathbf{p}d\mathbf{q}$$

$$q_{jj'}(\mathbf{k}')\delta(\mathbf{k}+\mathbf{k}') = \langle a_j(\mathbf{k}) a_{j'}(\mathbf{k}')\rangle$$

Towards a statistical description…



« Sea » of waves

$$\frac{\partial q_{jj'}\delta(k+k')}{\partial t} = \left\langle a_{j'}(\mathbf{k}')\frac{\partial a_j(\mathbf{k})}{\partial t}\right\rangle + \left\langle a_j(\mathbf{k})\frac{\partial a_{j'}(\mathbf{k}')}{\partial t}\right\rangle =$$

$$\varepsilon \int_{\mathbf{R}^6} \mathcal{H}^{\mathbf{kpq}}_{jmn}\langle a_m(\mathbf{p})a_n(\mathbf{q})a_{j'}(\mathbf{k}')\rangle e^{i\Omega_{k,pq}t}\delta_{k,pq}d\mathbf{p}d\mathbf{q}$$

$$+$$

$$\varepsilon \int_{\mathbf{R}^6} \mathcal{H}^{\mathbf{k'pq}}_{j'mn}\langle a_m(\mathbf{p})a_n(\mathbf{q})a_j(\mathbf{k})\rangle e^{i\Omega_{k',pq}t}\delta_{k',pq}d\mathbf{p}d\mathbf{q}\,.$$

$$\frac{\partial\langle a_j(\mathbf{k})a_{j'}(\mathbf{k}')a_{j''}(\mathbf{k}'')\rangle}{\partial t} =$$

$$\varepsilon \int_{\mathbf{R}^6} \mathcal{H}^{\mathbf{kpq}}_{jmn}\langle a_m(\mathbf{p})a_n(\mathbf{q})a_{j'}(\mathbf{k}')a_{j''}(\mathbf{k}'')\rangle e^{i\Omega_{k,pq}t}\delta_{k,pq}d\mathbf{p}d\mathbf{q}$$

$$+\varepsilon \int_{\mathbf{R}^6} \Big\{(\mathbf{k},j) \leftrightarrow (\mathbf{k}',j')\Big\}d\mathbf{p}d\mathbf{q}$$

$$+\varepsilon \int_{\mathbf{R}^6} \Big\{(\mathbf{k}",j") \leftrightarrow (\mathbf{k}',j')\Big\}d\mathbf{p}d\mathbf{q}\,,$$

$$\langle a_j(\mathbf{k})a_{j'}(\mathbf{k}')a_{j''}(\mathbf{k}'')\rangle = \tag{1.30}$$

$$\varepsilon \int_{\mathbf{R}^6} \mathcal{H}^{\mathbf{kpq}}_{jmn}\Big(\langle a_m(\mathbf{p})a_n(\mathbf{q})\rangle\langle a_{j'}(\mathbf{k}')a_{j''}(\mathbf{k}'')\rangle + \langle a_m(\mathbf{p})a_{j'}(\mathbf{k}')\rangle\langle a_n(\mathbf{q})a_{j''}(\mathbf{k}'')\rangle$$

$$+\langle a_m(\mathbf{p})a_{j''}(\mathbf{k}'')\rangle\langle a_n(\mathbf{q})a_{j'}(\mathbf{k}')\rangle\Big)\Delta(\Omega_{k,pq})\delta_{k,pq}d\mathbf{p}d\mathbf{q}$$

$$+\varepsilon \int_{\mathbf{R}^6} \Big\{(\mathbf{k},j) \leftrightarrow (\mathbf{k}',j')\Big\}d\mathbf{p}d\mathbf{q} + \varepsilon \int_{\mathbf{R}^6} \Big\{(\mathbf{k}",j") \leftrightarrow (\mathbf{k}',j')\Big\}d\mathbf{p}d\mathbf{q}\,,$$

where

$$\Delta(\Omega_{k,pq}) = \int_0^{t\gg 1/\omega} e^{i\Omega_{k,pq}t'}dt' = \frac{e^{i\Omega_{k,pq}t}-1}{i\Omega_{k,pq}}\,. \qquad \Delta(x) \to \pi\delta(x) + i\mathcal{P}(1/x)$$

Asymptotic closure: only resonance terms survive

$$\boxed{\begin{aligned}\frac{\partial q_{jj'}(\mathbf{k})}{\partial t} &= 4\pi\varepsilon^2 \int_{\mathbf{R}^6} \delta_{k,pq}\delta(\Omega_{k,pq})\mathcal{H}^{\mathbf{kpq}}_{jmn} \qquad (1.35)\\ &\Big(\mathcal{H}^{\mathbf{p-q-k}}_{mrs}q_{rn}(\mathbf{q})q_{j's}(\mathbf{k}) + \mathcal{H}^{\mathbf{q-pk}}_{nrs}q_{rm}(\mathbf{p})q_{j's}(\mathbf{k}) + \mathcal{H}^{\mathbf{-k-p-q}}_{j'rs}q_{rm}(\mathbf{p})q_{sn}(\mathbf{q})\Big)d\mathbf{p}d\mathbf{q}\,.\end{aligned}}$$

$\tau_w << \tau_{eddy}$

Weak whistler / KAW turbulence

$$k_\perp \gg k_\|$$

$$\partial_t \left\{ \begin{matrix} E_k \\ H_k \end{matrix} \right\} = \frac{\epsilon^2}{16} \sum_{s s_p s_q} \int \frac{s_p\, p_\perp k_\| p_\|}{q_\perp} \left(\frac{s_q q_\perp - s_p p_\perp}{k_\|} \right)^2 (s k_\perp + s_p p_\perp + s_q q_\perp)^2 \sin\theta_q$$

$$\left\{ \begin{matrix} s k_\perp \left[E_q (p_\perp E_k - k_\perp E_p)/(k_\perp p_\perp q_\perp) + s_q\, H_q\, (s H_k - s_p H_p) \right] \\ E_q (s\, H_k - s_p H_p)/q_\perp + s_q\, H_q (p_\perp E_k - k_\perp E_p)/(k_\perp p_\perp) \end{matrix} \right\}$$

$$\delta(k_\| + p_\| + q_\|)\, \delta(s k_\perp k_\| + s_p p_\perp p_\| + s_q q_\perp q_\|)\, dp_\perp dq_\perp dp_\| dq_\| \,.$$

<u>Exact solution at constant magnetic energy flux:</u> [SG & Bhattacharjee, PoP, 2003]

$$\begin{matrix} E_k \sim k_\perp^n |k_\||^m \\ H_k \sim k_\perp^{\tilde n} |k_\||^{\tilde m} \end{matrix} \quad \xrightarrow[\text{transformation}]{\text{Zakharov}} \quad \boxed{\begin{matrix} n = -5/2,\ m = -1/2 \\ \tilde n = -7/2 \text{ and } \tilde m = -1/2 \end{matrix}}$$

Positive flux $\Rightarrow$ direct $\perp$ cascade

Can be predicted by classical phenomenology

Weak whistler / KAW turbulence

$$k_\perp \gg k_\parallel$$

$$\partial_t \begin{Bmatrix} E_k \\ H_k \end{Bmatrix} = \frac{\epsilon^2}{16} \sum_{s s_p s_q} \int \frac{s_p\, p_\perp k_\parallel p_\parallel}{q_\perp} \left(\frac{s_q q_\perp - s_p p_\perp}{k_\parallel} \right)^2 (s k_\perp + s_p p_\perp + s_q q_\perp)^2 \sin\theta_q$$

$$\begin{Bmatrix} s k_\perp \left[E_q (p_\perp E_k - k_\perp E_p)/(k_\perp p_\perp q_\perp) + s_q\, H_q\, (s H_k - s_p H_p) \right] \\ E_q (s\, H_k - s_p H_p)/q_\perp + s_q\, H_q (p_\perp E_k - k_\perp E_p)/(k_\perp p_\perp) \end{Bmatrix}$$

$$\delta(k_\parallel + p_\parallel + q_\parallel)\, \delta(s k_\perp k_\parallel + s_p p_\perp p_\parallel + s_q q_\perp q_\parallel)\, dp_\perp dq_\perp dp_\parallel dq_\parallel \,.$$

Exact solution at constant magnetic helicity flux: [SG & Meyrand, JPP, 2015]

$$\begin{matrix} E_k \sim k_\perp^n |k_\parallel|^m \\ H_k \sim k_\perp^{\tilde{n}} |k_\parallel|^{\tilde{m}} \end{matrix} \xrightarrow[\text{transformation}]{\text{Zakharov}} \boxed{\begin{matrix} n + \tilde{n} = -6 \\ m + \tilde{m} = -1 \end{matrix}}$$

Locality conditions: $$\begin{matrix} -3 < n + m < -2 \,, \\ -4 < \tilde{n} + \tilde{m} < -3 \,. \end{matrix}$$

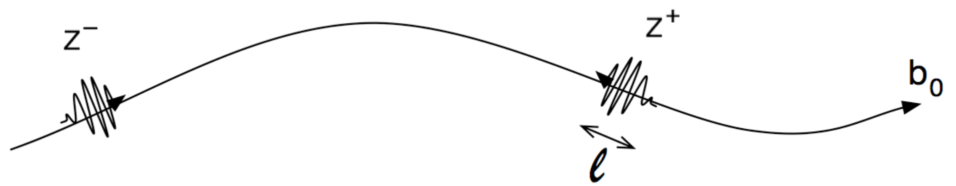
Negative flux $\Rightarrow$ inverse $\perp$ cascade

Non-trivial solutions: classical phenomenology does not work !

Classical WT phenomenology: MHD case

$$\mathbf{z}^{\pm} \equiv \mathbf{u} \pm \mathbf{b}$$

$$\frac{\partial \mathbf{z}^{\pm}}{\partial t} \mp \mathbf{b}_0 \cdot \nabla \mathbf{z}^{\pm} + \underbrace{\mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm}}_{\text{collision}} = -\nabla P_* + \nu_+ \Delta \mathbf{z}^{\pm} + \nu_- \Delta \mathbf{z}^{\mp}$$



$$z^+ \sim z^- \sim z$$

$\tau_A \sim \frac{\ell}{b_0}$ duration of a collision = **Alfvén time**

$$\Longrightarrow \quad z_\ell(t+\tau_A) \sim z_\ell(t) + \tau_A \frac{\partial z_\ell}{\partial t} \sim z_\ell(t) + \tau_A \frac{z_\ell^2}{\ell}$$

Therefore, the deformation of the wave-packet after one collision is :

$$\Delta_1 z_\ell \sim \tau_A \frac{z_\ell^2}{\ell} \,.$$

This deformation will grow with time and for N stochastic collisions the cumulative effect can be evaluated in the same manner as a random walk :

$$\sum_{i=1}^{N} \Delta_i z_\ell \sim \tau_A \frac{z_\ell^2}{\ell} \sqrt{\frac{t}{\tau_A}} \,. \qquad (12.23)$$

Cumulative distorsion of order one defines τ_{tr} : $z_\ell \sim \sum_1^N \Delta_i z_\ell \sim \tau_A \frac{z_\ell^2}{\ell} \sqrt{\frac{\tau_{tr}}{\tau_A}}$

$$\tau_{eddy} \sim \ell / u_\ell \qquad \tau_{tr} \sim \frac{1}{\tau_A} \frac{\ell^2}{z_\ell^2} \sim \frac{\tau_{eddy}^2}{\tau_A} \Longrightarrow \varepsilon \sim \frac{z_\ell^2}{\tau_{eddy}^2 / \tau_A} \sim \frac{z_\ell^4}{\ell b_0} \sim \frac{E^2(k) k^3}{b_0}$$

IK SPECTRUM

$$\Longrightarrow \boxed{E^z(k) = C_{IK} \sqrt{\varepsilon b_0}\, k^{-3/2}}$$

[Iroshnikov, SA, 1964; Kraichnan, PoF, 1965]

balance
turbulence

Same phenomenology as IK but with anisotropy

$$\boxed{E(k_\perp, k_\parallel) \sim \sqrt{\varepsilon b_0}\, k_\perp^{-2} k_\parallel^{-1/2}}$$

[SG et al., JPP, 2000]

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Same phenomenology as IK but with anisotropy

$$\boxed{E(k_\perp, k_\parallel) \sim \sqrt{\varepsilon b_0}\, k_\perp^{-2} \cancel{k_\parallel^{-1/2}}}$$

[SG et al., JPP, 2000]

+ Resonance condition

Classical WT phenomenology: R-EMHD case

$$\tau_{eddy} \sim \ell_\perp^2/(d_i b_\ell) \qquad \tau_W \sim 1/(k_\| k_\perp d_i b_0)$$

$$\tau_{tr} \sim \frac{\tau_{eddy}^2}{\tau_W}$$

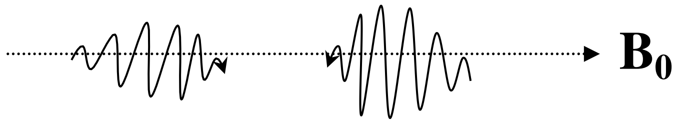
$$\Longrightarrow \qquad \varepsilon \sim \frac{b_\ell^2}{\tau_{tr}} \sim \frac{d_i b_\ell^4 k_\perp^3}{k_\| b_0}$$

$$\Longrightarrow \qquad \boxed{E^b(k_\perp, k_\|) \sim k_\perp^{-5/2} k_\|^{-1/2}}$$

[SG & Bhattacharjee, PoP, 2003]

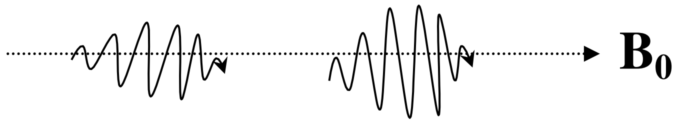
New WT phenomenology

Balanced case



Hm = 0

Imbalanced case



Hm ≠ 0

$$k_\perp \gg k_\parallel \qquad V_g \sim d_i k_\perp b_0 \qquad \tau_{coll} \sim \ell_\parallel / \Delta V_g$$

Collision between two wave-packets travelling in the same direction

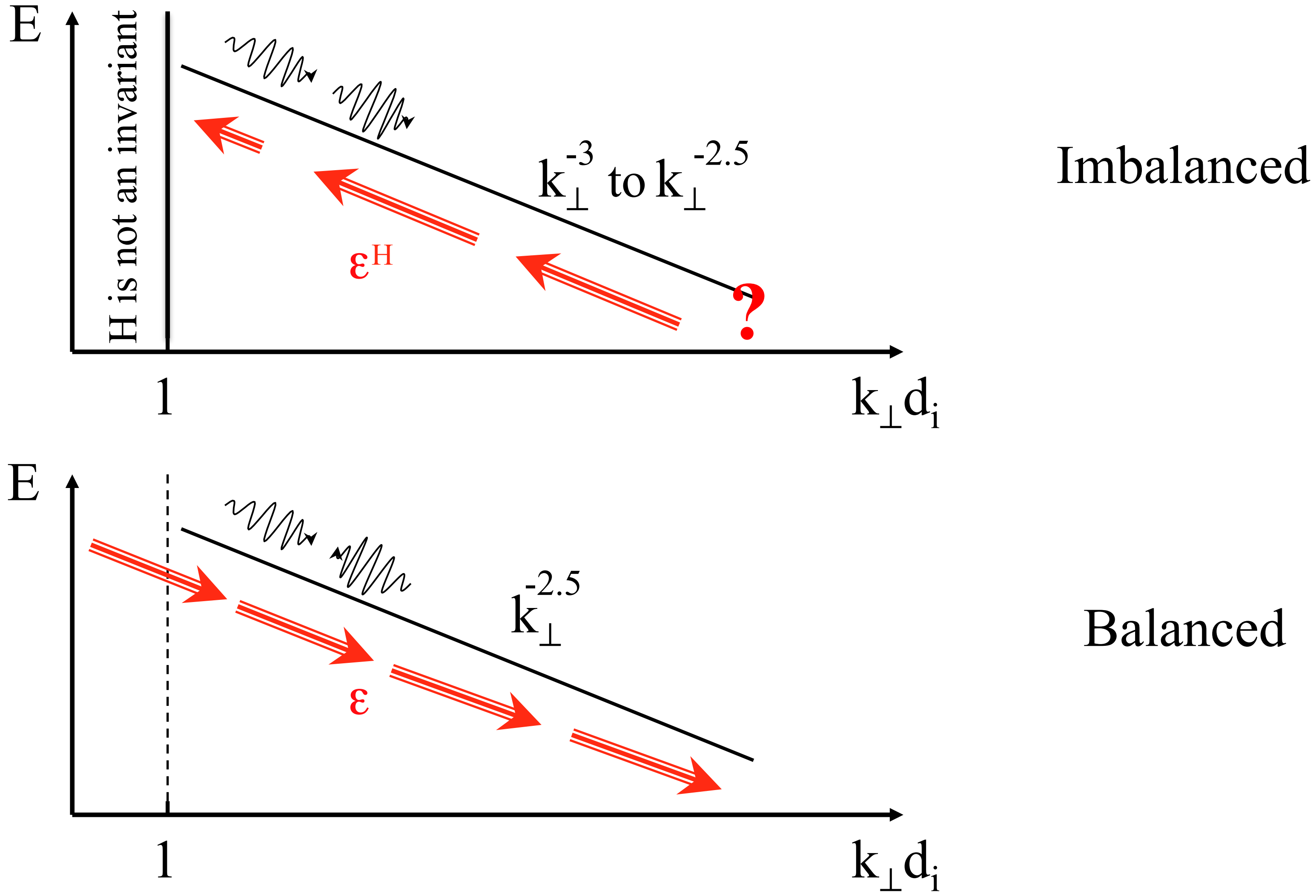
$$\tau_{eddy} \sim \ell_\perp^2 / (d_i b_\ell)$$

$$\tau_{tr} \sim \frac{\tau_{eddy}^2}{\tau_{coll}} \Longrightarrow \varepsilon^H \sim \frac{H_\ell}{\tau_{tr}} \sim \frac{d_i H_\ell E_\ell k_\perp^4}{\underbrace{\Delta k_\perp}k_\parallel b_0}$$

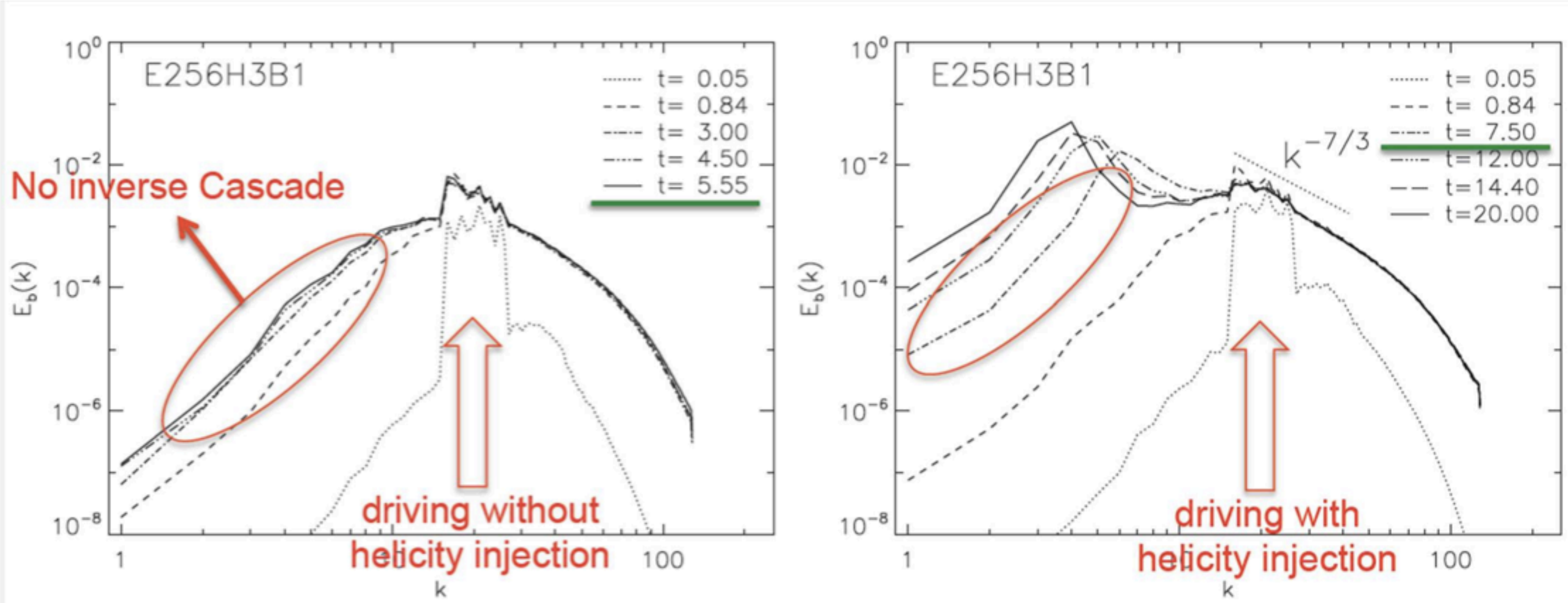
Local cascade: it is constant

$$\Longrightarrow \boxed{H(k_\perp, k_\parallel) E(k_\perp, k_\parallel) \sim \left(\frac{\varepsilon^H \Delta k_\perp b_0}{d_i} \right) k_\perp^{-6} k_\parallel^{-1}}$$

Magnetic helicity flux



Magnetic helicity flux



[Kim & Cho, ApJ, 2015]

There is no hypo-viscosity => impossible to reach a stationary solution

Imbalanced turbulence

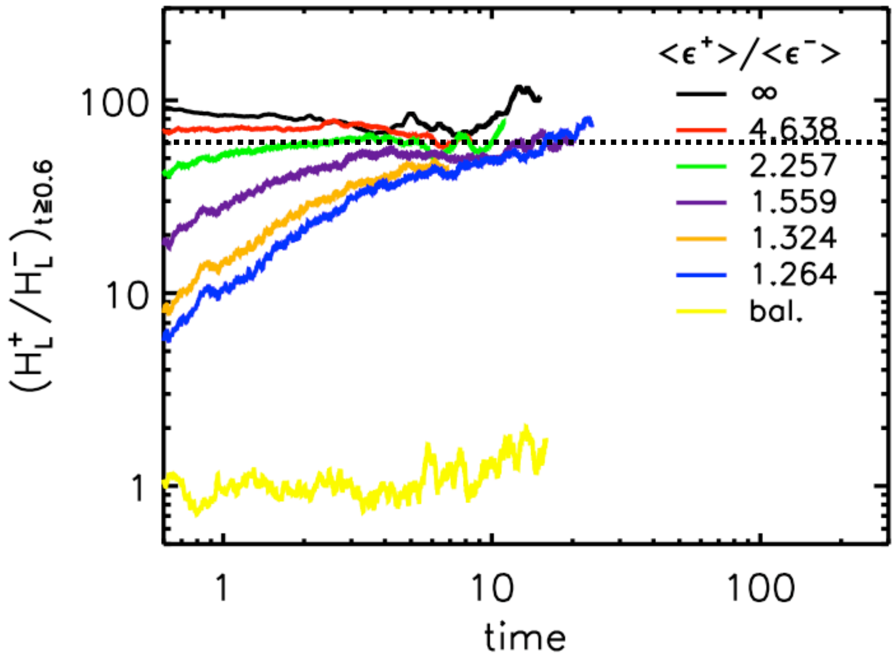
✓ Pure imbalanced R-EMHD turbulence is impossible [Cho, PRL, 2011]

⟶ sub-dominant (counter-propagating) wave-packets are created

✓ Saturation of the amount of sub-dominant wave-packets

$$H_L^+ \equiv \sum_1^{k_f/1.3} E_h^+(k) = \sum_1^{k_f/1.3} E_b^+(k)/k \text{ and}$$

$$H_L^- \equiv \sum_1^{k_f/1.3} E_h^-(k) = \sum_1^{k_f/1.3} E_b^-(k)/k.$$



[Kim & Cho, ApJ, 2015]