Power law estimation

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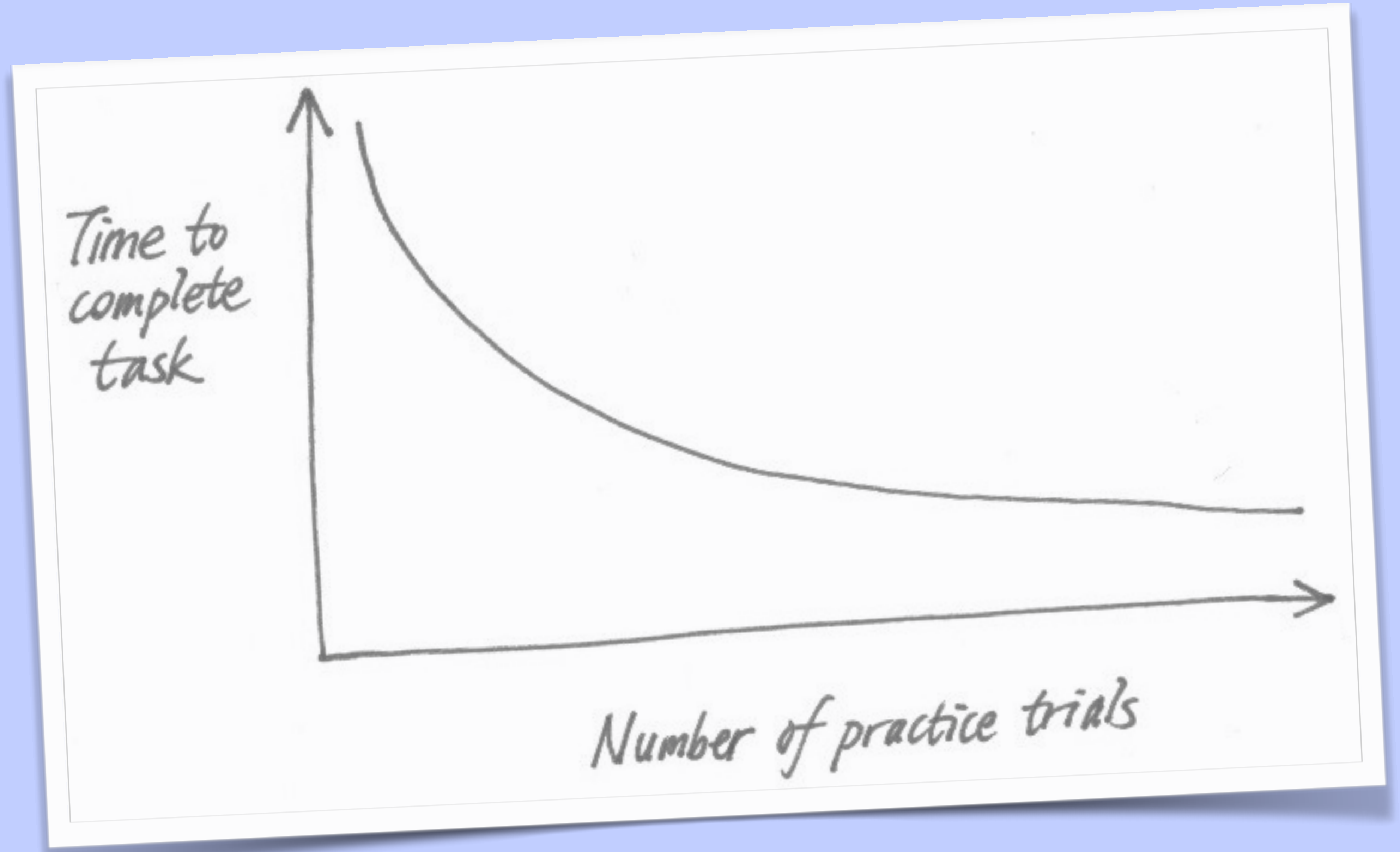


Why look for power laws ?

- Basic property : scale invariance $y(\lambda t) \sim \lambda^k y(t)$

In the spectral domain $\tilde{y}(f) \propto f^{-\beta}$

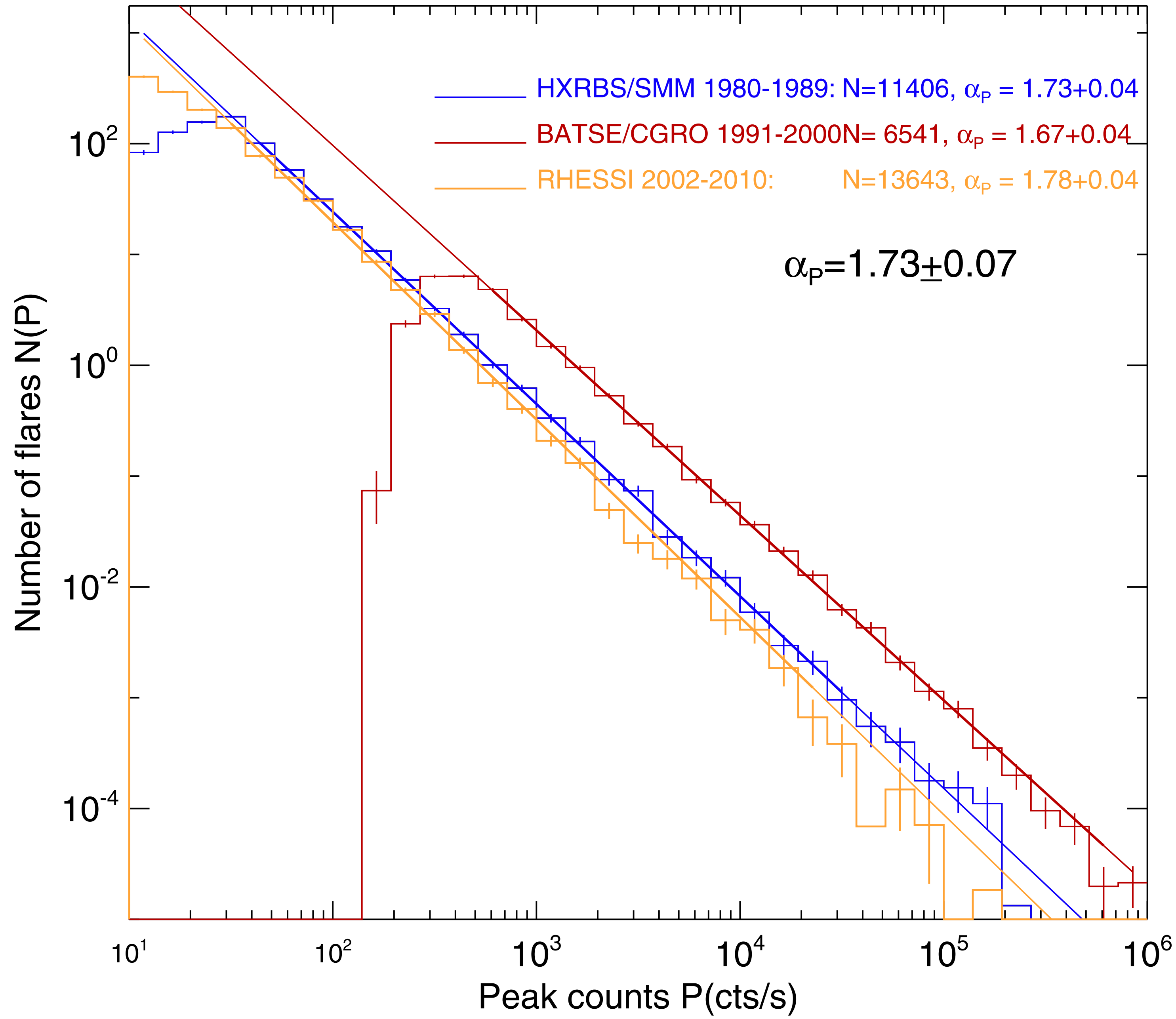
Why look for power laws ?

- Lots of fascinating properties
  - scale-invariance (no characteristic scale)
  - large deviations (rare events)
  - turbulence (intermittency, energy cascades, ...)

- Self-similarity is ubiquitous in plasmas
  - solar flare energy distribution
  - plasma turbulence
  - bursty bulk flows in the magnetosphere
  - photospheric magnetic field
  - ...

Example: flare energy distribution

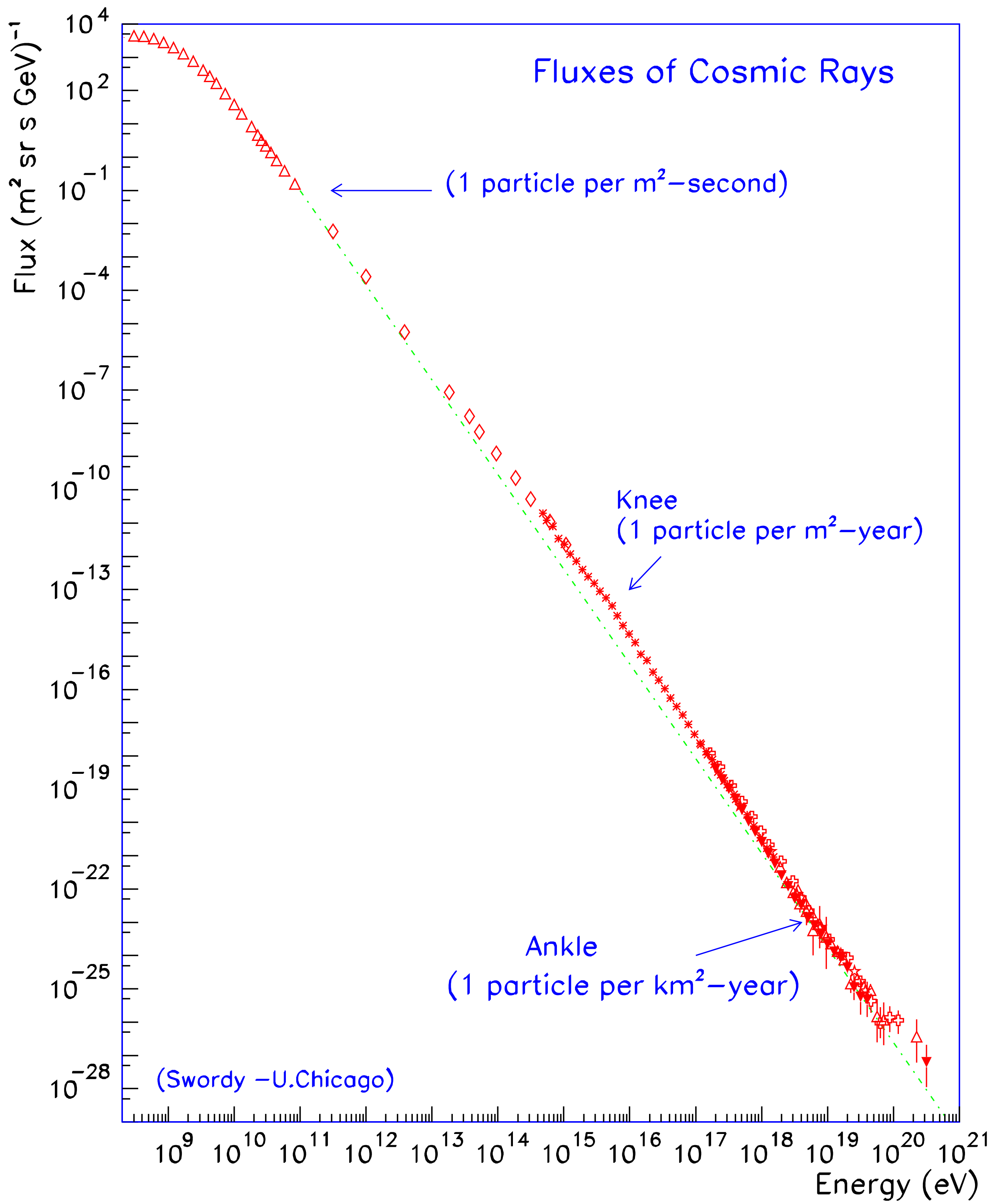
Occurrence frequency distributions of hard X-ray peak count rates from 3 instruments.



Aschwanden et al., 2014

Example: cosmic ray spectrum

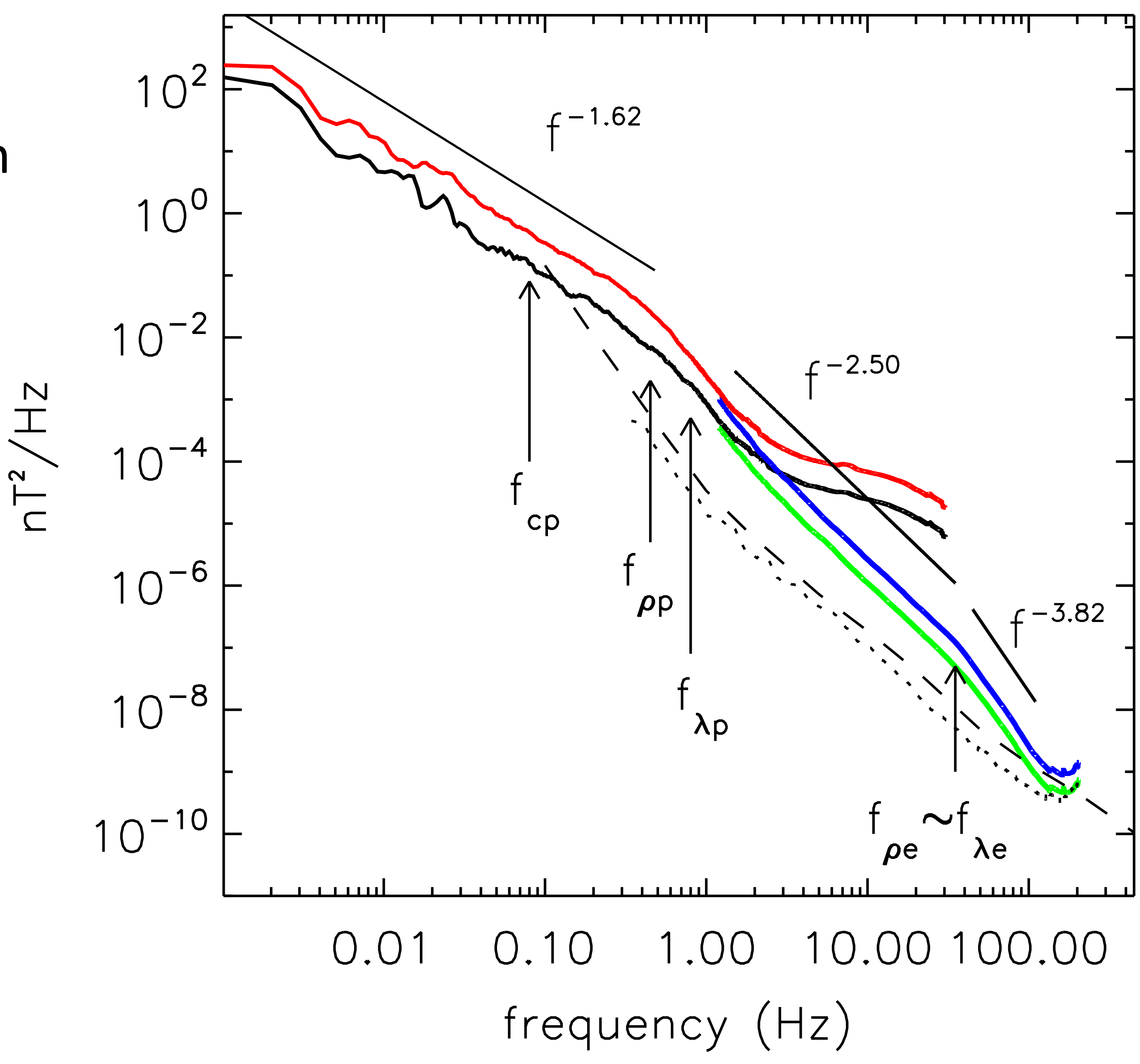
Cosmic ray spectrum between 1 GeV and 1 ZeV



Aschwanden et al., 2014

Example: plasma turbulence

Power spectral density of the magnetic field in solar wind turbulence

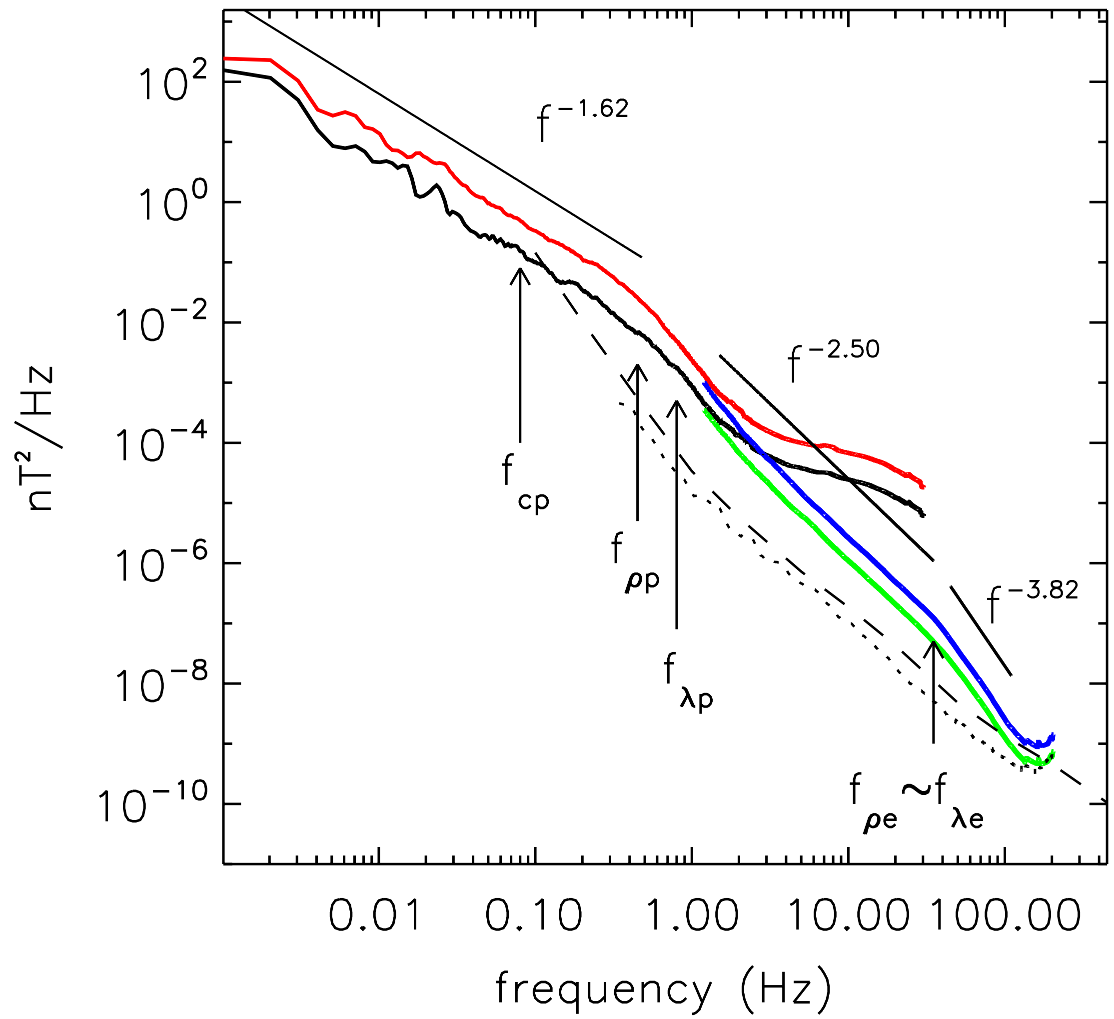


Sahraoui et al., 2009

Why look for power laws ?

- The power law itself is **totally uninteresting**

- What physical process gave rise to it ?
  - where does the scaling start ?
  - where does the scaling end ?
  - what is its slope ?



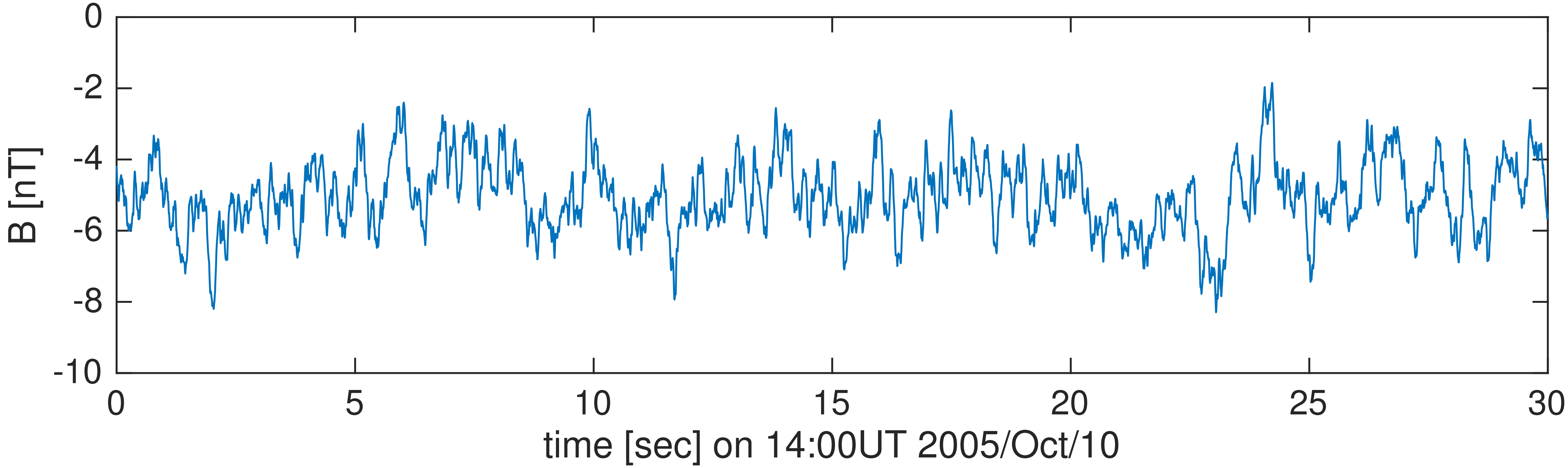
Example :
solar wind magnetic field recorded by Anik F1R

Solar wind observations

- Anik F1R (2005-today): Canadian geostationary satellite
  - experimental magnetic field sensor (fluxgate, 1 component)
  - high cadence : $<$ 200 samples / second
  - some observations made in the solar wind

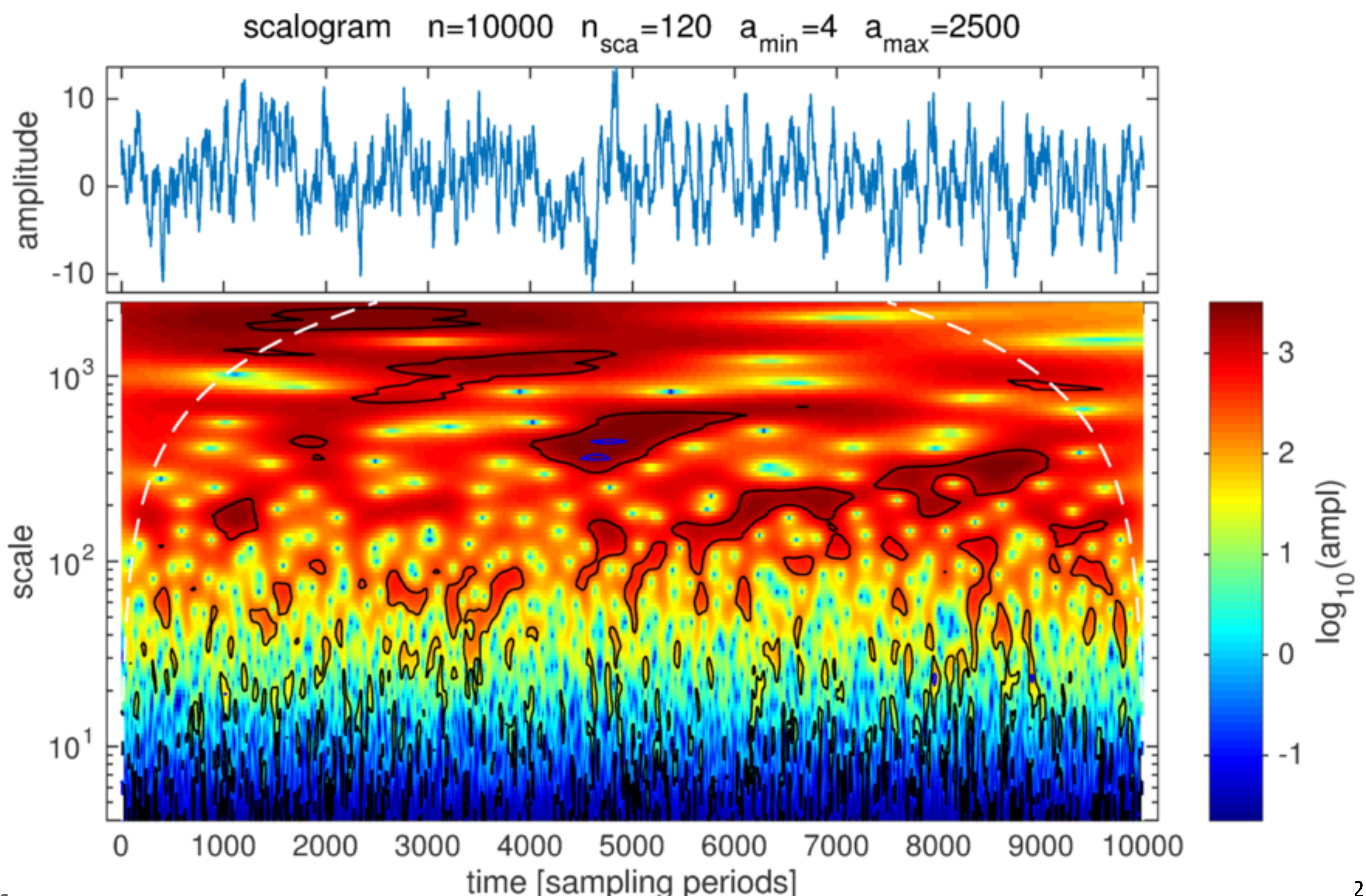
Magnetic field observations

- Raw data (essentially B_y)



Scalogram

Modulus of continuous wavelet transform (Morlet wavelets)



Intermittency ?

- To identify intermittent bursts of activity : estimate the **local intermittency measure** [Farge, 1992; 1996]

$$B(t) \quad \longrightarrow \quad \tilde{B}(a,\tau) \quad \longrightarrow \quad \mathrm{LIM}(a,\tau) = \frac{|\tilde{B}(a,\tau)|^2}{\langle |\tilde{B}(a,\tau')|^2 \rangle_{\tau'}}$$

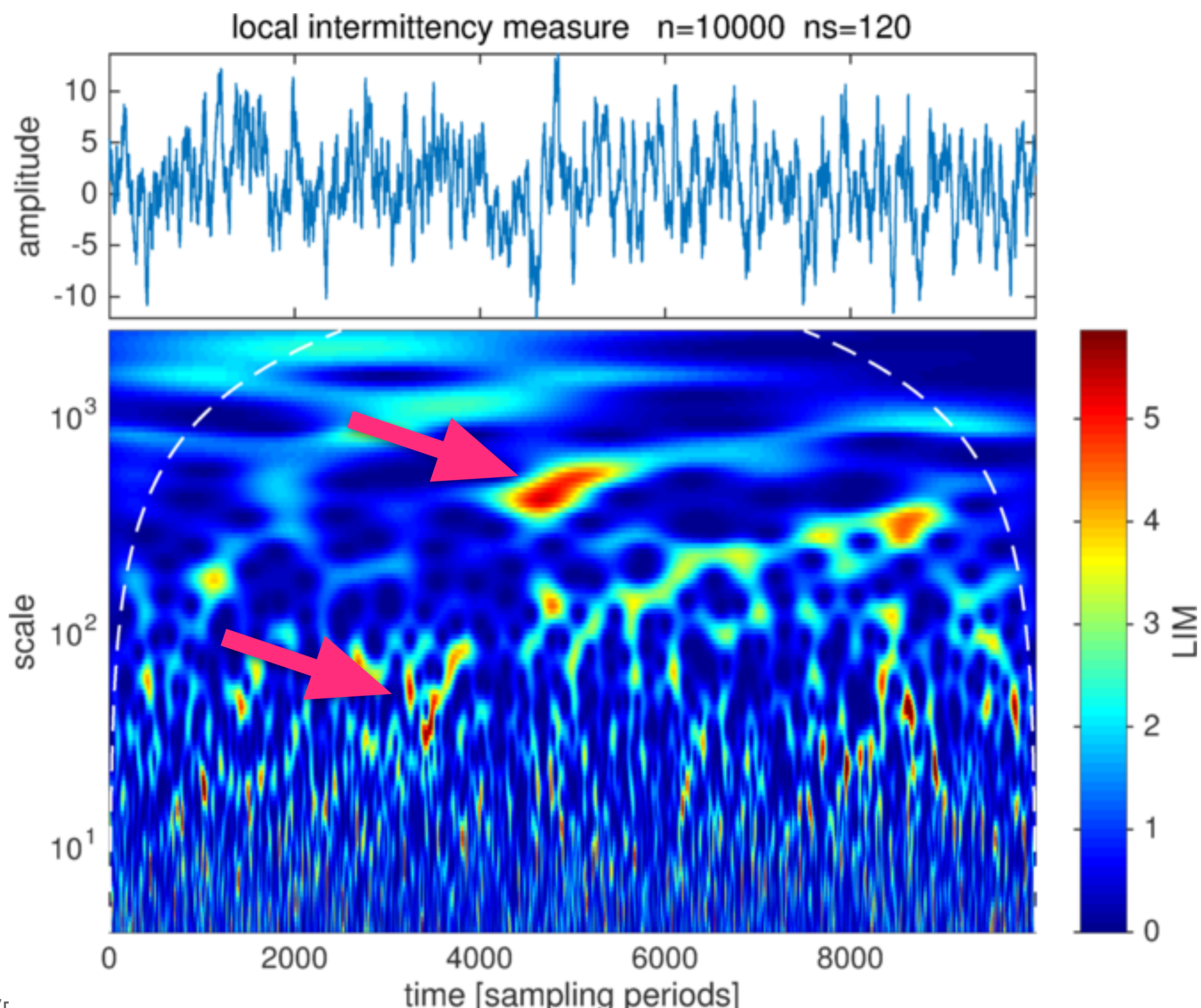
wavelet transform at scale a and time τ

local intermittency measure

- Large LIM = local concentration of magnetic energy at scale a and time τ

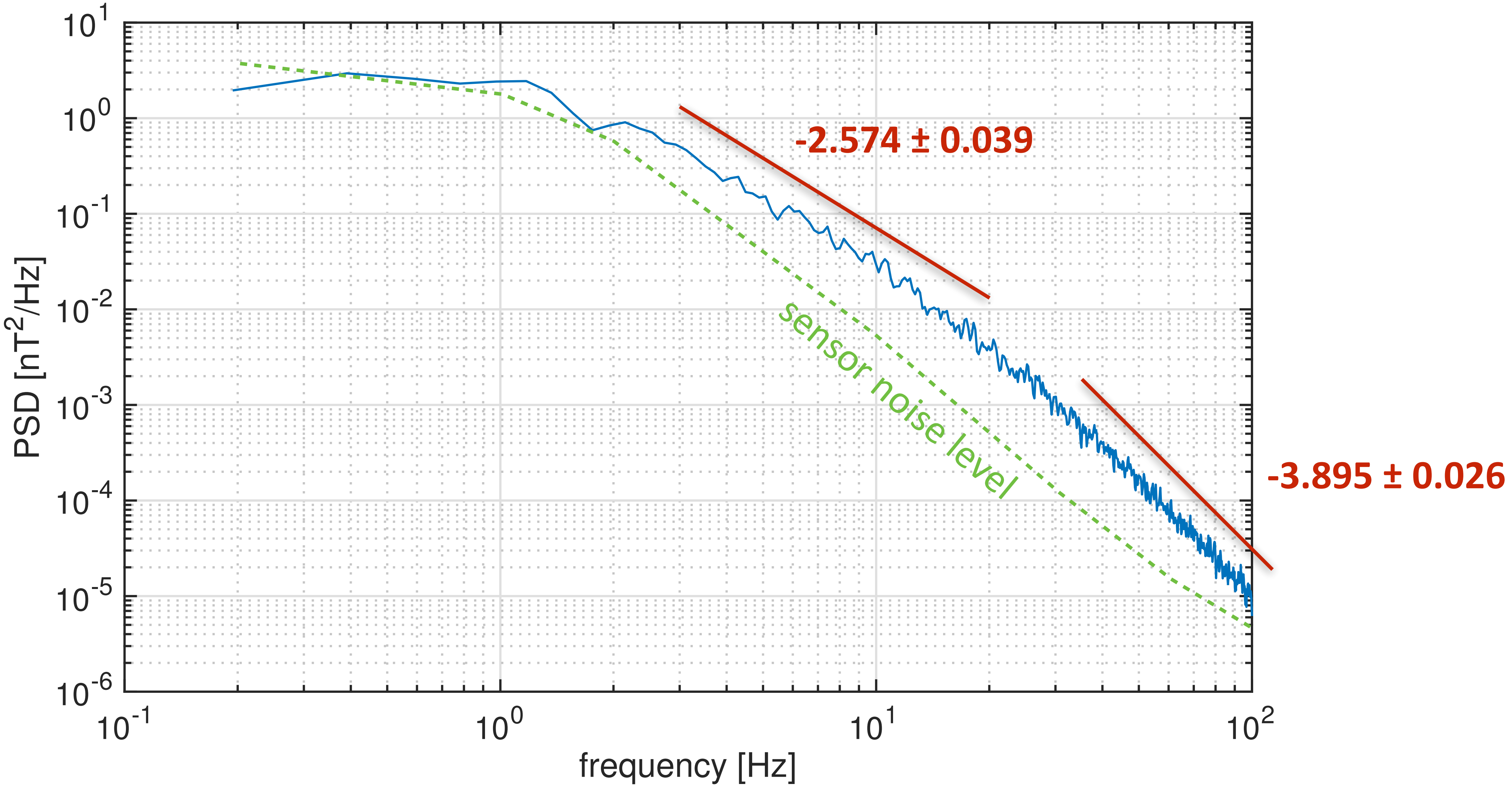
Scalogram

Local intermittency measure



Power spectral density

- Clear evidence for a double power law, with a break



All of this is just a dream

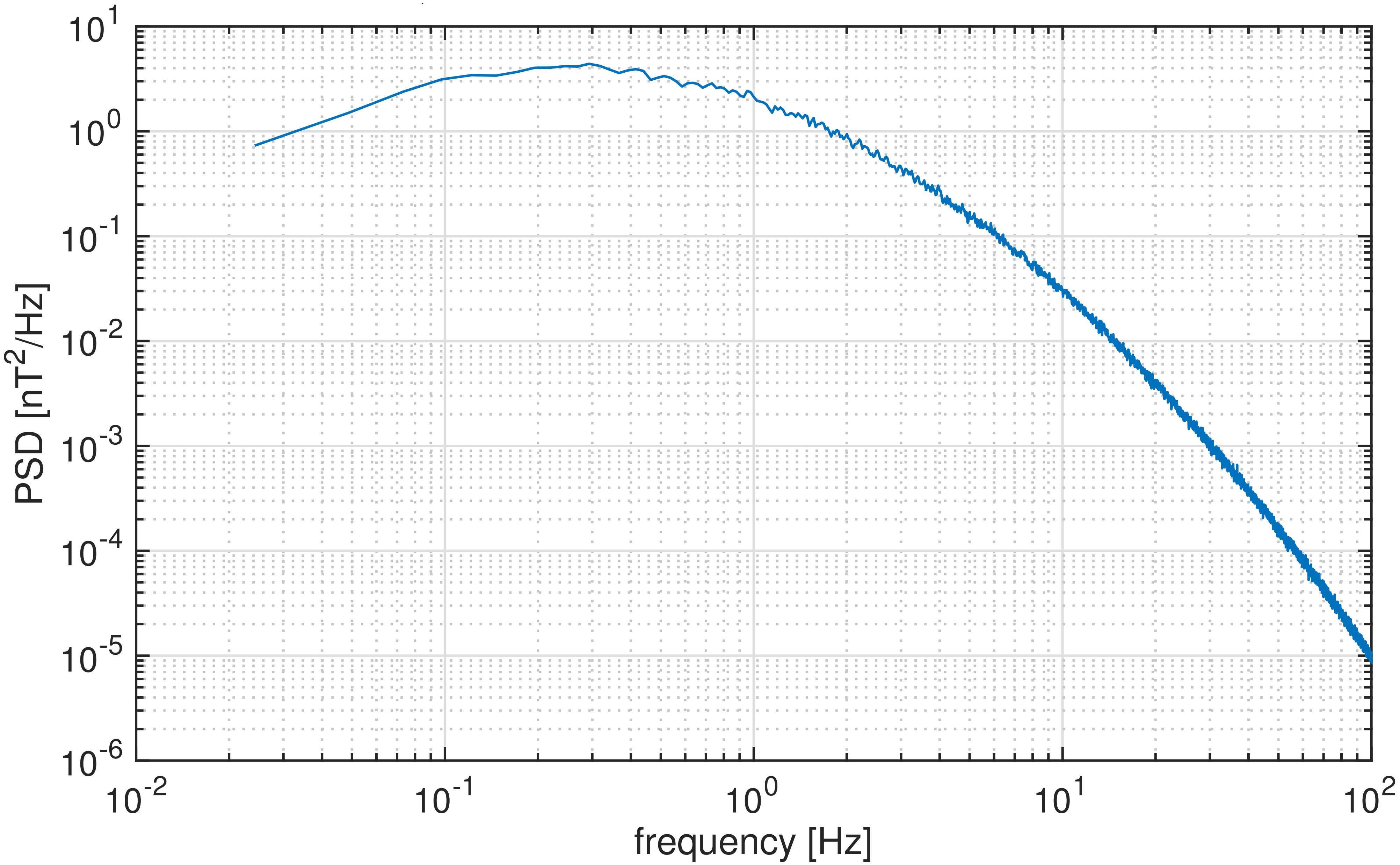
Disclaimer

- My record was generated by a simple stochastic (autoregressive) process:
  no intermittency or self-similarity whatsoever!

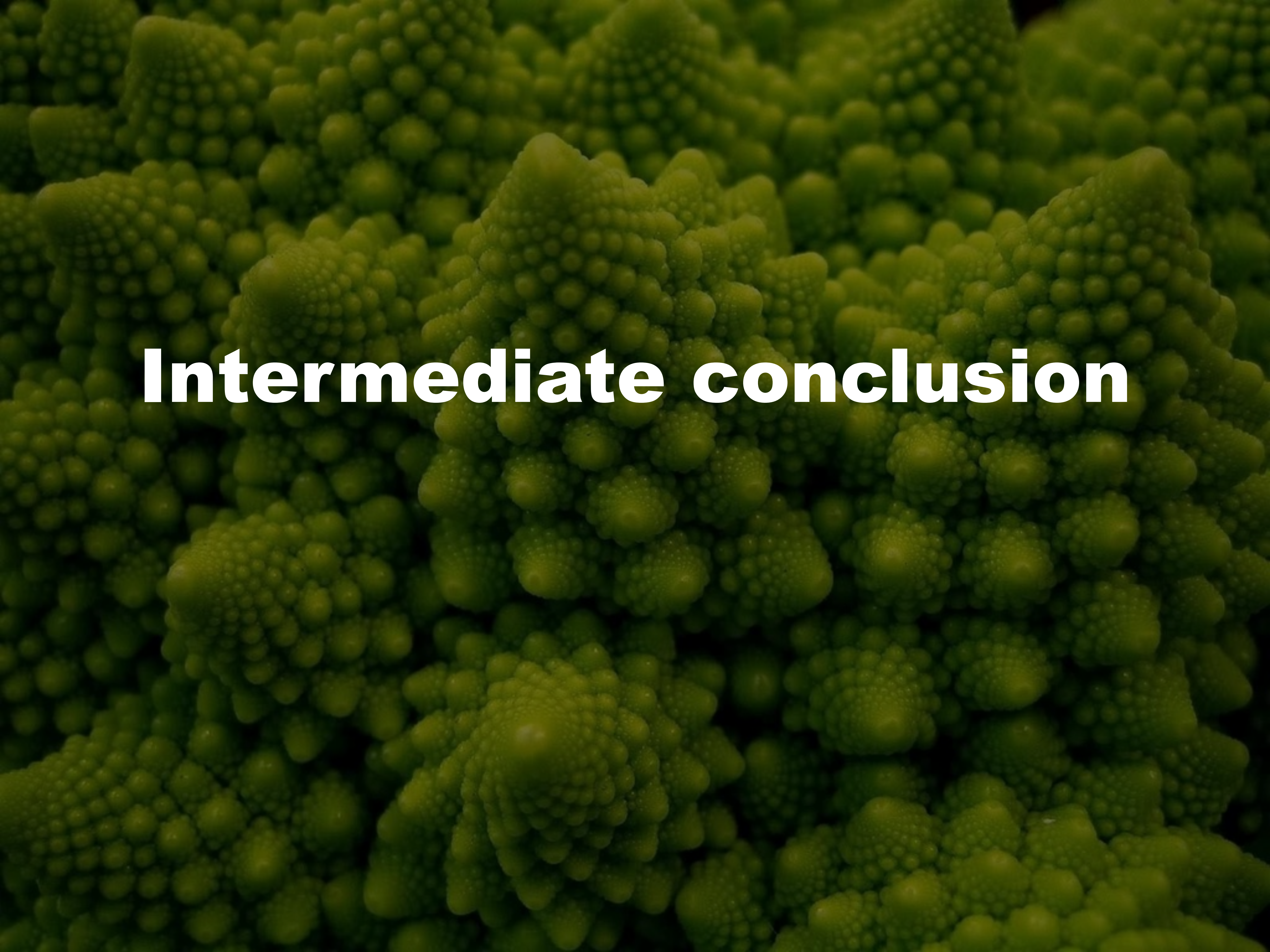
- The true power spectral density is a log-normal one:
  no power laws here!

Disclaimer

- The true power spectral density $\tilde{y}(f) = P_0 \exp\left(-\frac{(f-f_0)^2}{\Delta f^2}\right)$



Intermediate conclusion



Intermediate conclusion

We are completely fooled by our frenzy to detect straight lines

The problem

relatively easy

1) Estimate the power spectral density $|\tilde{y}(f)|^2 \propto f^{-\alpha}$

$\downarrow$

need a good estimator of the slope α

2) Estimate the slope (spectral index) α

$\downarrow$

test against a null hypothesis (« no power law model »)

3) Prove that the power law model is indeed appropriate

difficult

2 cases occur in practice

case 1 : **data = series of events**:

their probability density function follows a power law

(e.g. solar flare distribution)

case 2 : **data = time series**:

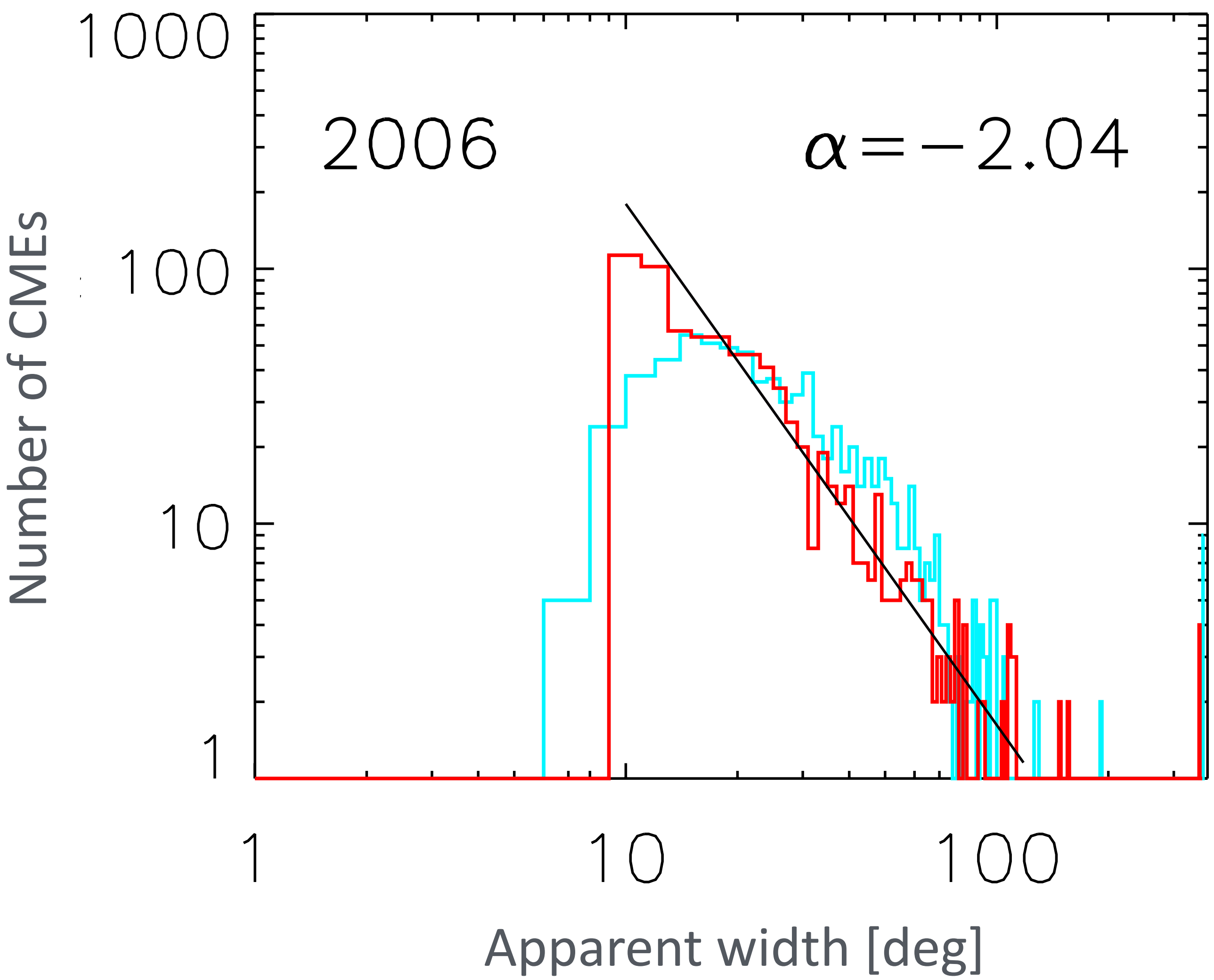
their power spectral density follows a power law

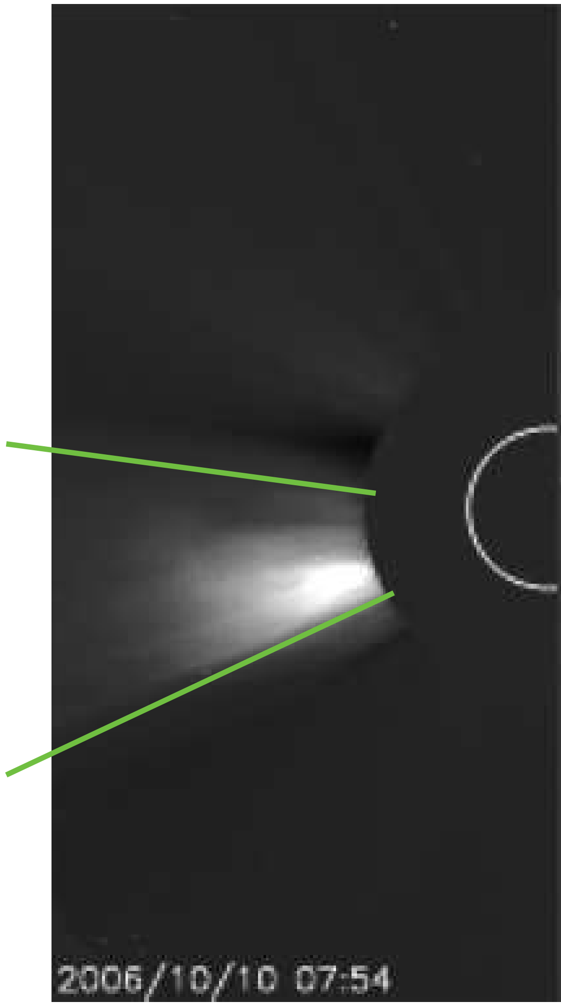
(e.g. E or B field record)

Case 1

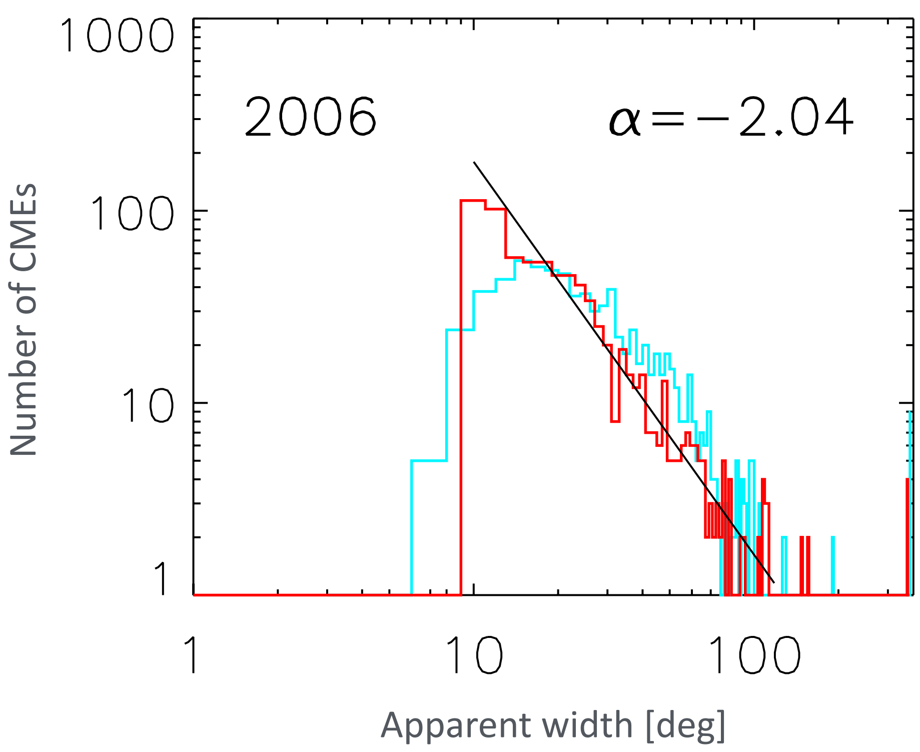
Case 1 : the data consist of events

- Example : number of Coronal Mass Ejections (CMEs) per apparent angular width [Robbrecht et al., ApJ 2009]





Case 1 : the data consist of events



- Classical approach
  - bin events to get a histogram (what bin size ?)
  - fit by least-squares a line to log(# of events) vs log(width)

This approach is full of flaws and will give biased and noisy estimates. Avoid it!

Case 1 : the data consist of events

- A much better estimator (unbiased, and low variance)
  [Clauset et al., SIAM review 2006]

  - select the range over which a power law will be fitted

$$y_{min} \leq y_i \leq y_{max}$$

  - use the maximum likelihood estimator of the slope

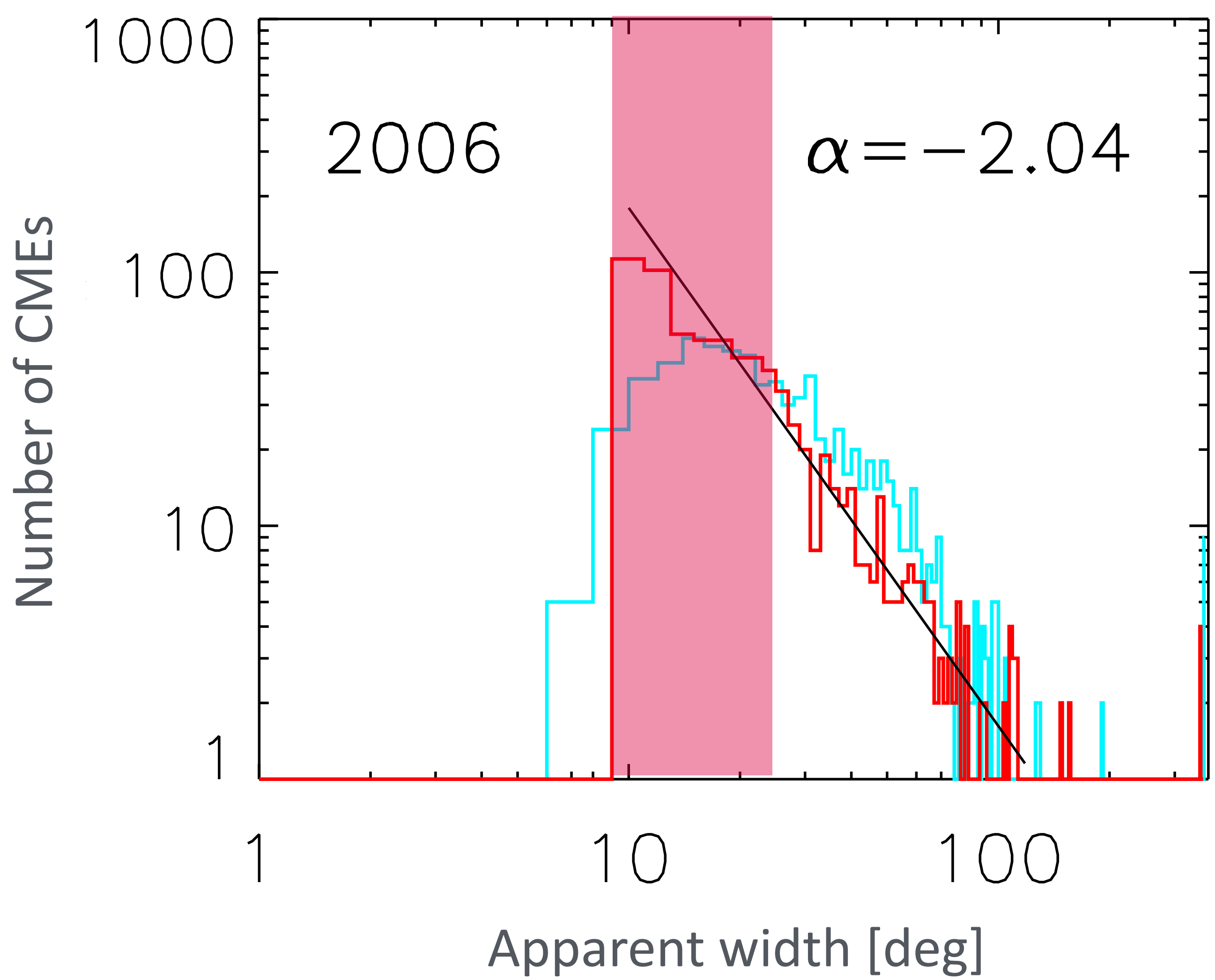
$$\hat{\alpha} = 1 + n \left(\sum_{i=1}^{n} \log \frac{y_i}{y_{min}} \right)^{-1}$$

  - whose standard error is

$$\sigma_{\hat{\alpha}} = \frac{\hat{\alpha} - 1}{\sqrt{n}} + \mathcal{O}(1/n)$$

Case 1 : the data consist of events

- Where does the lower bound (y_{min}) occur ?



Underestimating the lower bound is far worse than overestimating it

Strategy: select various lower bounds, fit power laws, and determine which one matches the data best

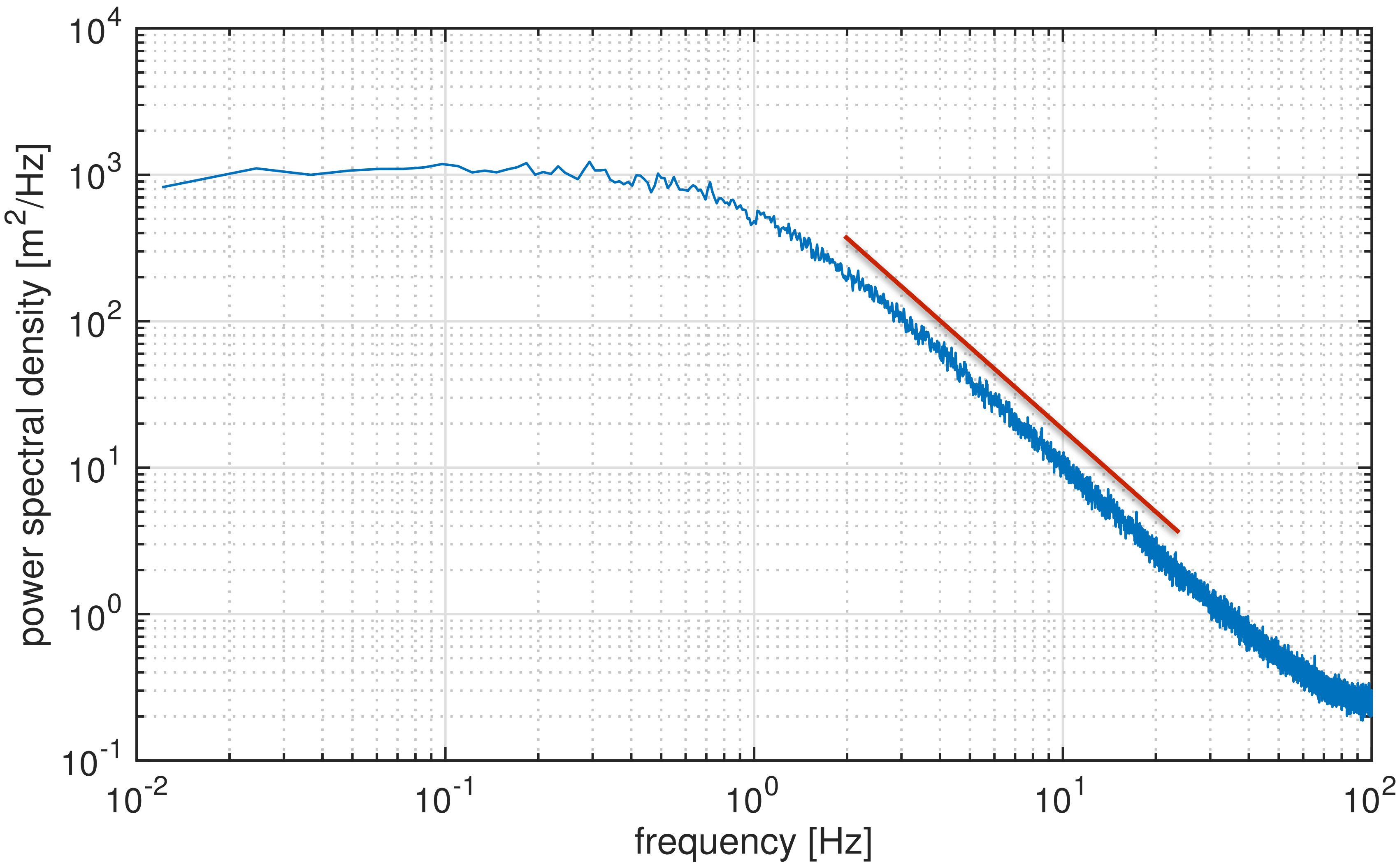

Case 2

Case 2 : time series

- 2 important steps for time series

  1) Estimate the spectral content of the time series

  2) Fit power law to the power spectral density



Case 2: time series

- Classical Fourier analysis is not optimal:
  a time series with self-similarity should be projected on self-similar basis functions ($\neq$ sines)

- Discrete wavelet transform is more appropriate:
  self-similar by construction

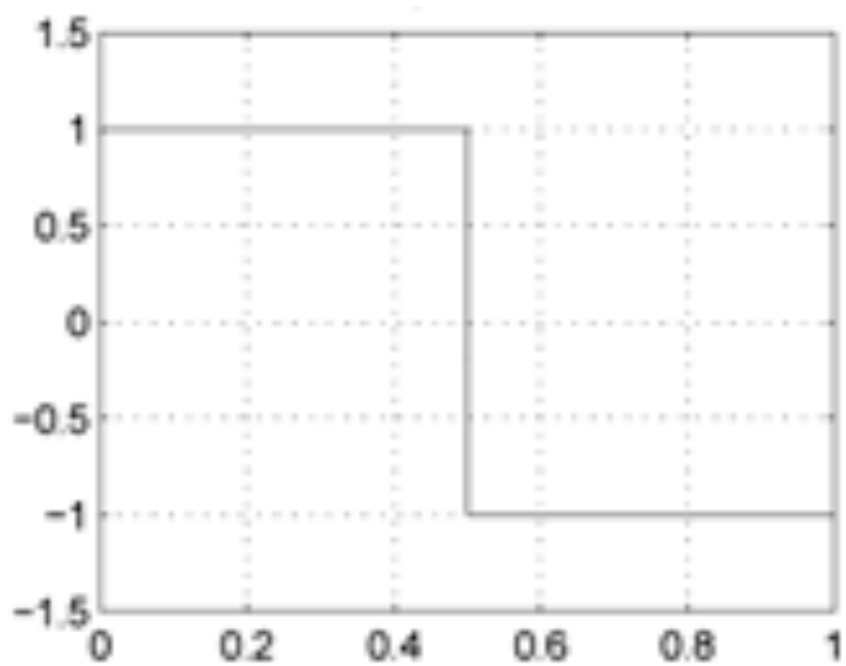
$$\tilde{y}_{j,k} = \int_{\mathbb{R}} y(t)\, \psi_{j,k}(t)\, \mathrm{d}t, \quad j,k \in \mathbb{Z}$$

where $$\psi_{j,k}(t) = \frac{1}{2^{j/2}} \psi\left(\frac{t}{2^j} - k\right), \quad j,k \in \mathbb{Z}$$

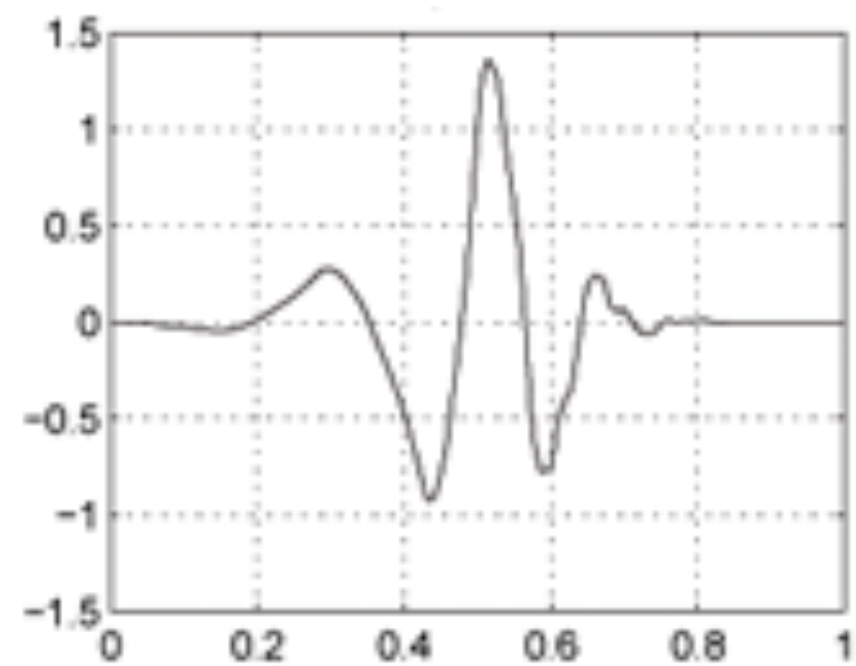
$\psi(\ )$ is the mother wavelet

Case 2: time series

- Lots of different mother wavelets around



Haar wavelet (db1)
1 vanishing moment



Daubechies db4 wavelet
4 vanishing moments

- The number of vanishing moments is a crucial parameter

Case 2: time series

Fourier

- Compute transform

$$\tilde{y}(f) = \int y(t)\, e^{-j2\pi f t}\, \mathrm{d}t$$

- Power spectral density

$$S(f) = |\tilde{y}(f)|^2$$

- can be estimated at all frequencies
- nearby values of S(f) are not independent

Discrete wavelet transform

- Compute transform

$$\tilde{y}_{j,k} = \int_{\mathbb{R}} y(t)\, \psi_{j,k}(t)\, \mathrm{d}t$$

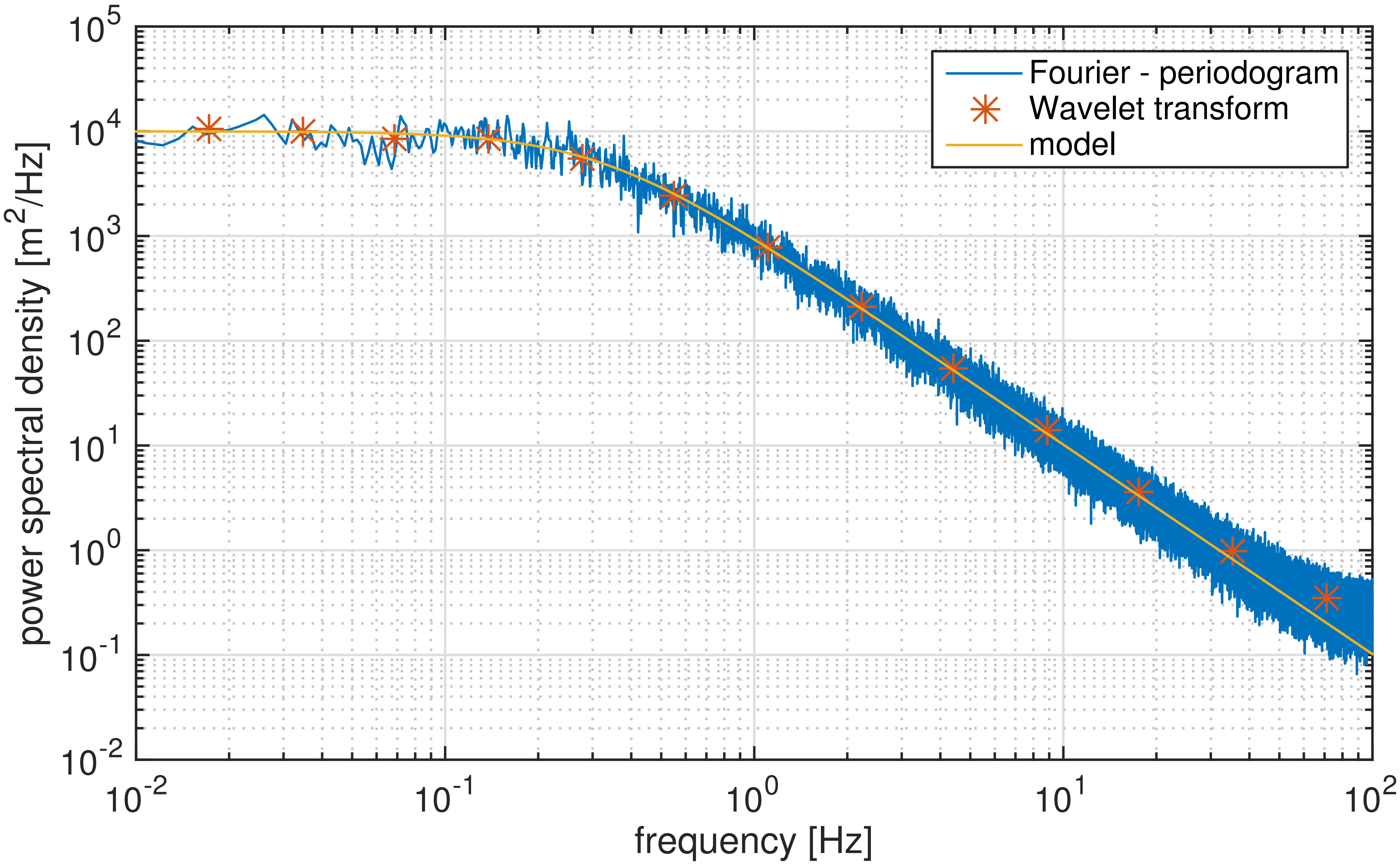
- Power spectral density

$$S_j = \langle |\tilde{y}_{j,k}|^2 \rangle_k$$

- can be estimated per octave only ($\text{scale}_j = 1, 2, 4, 8, \ldots$)
- wavelet coefficients at nearby scales/positions are independent (very useful)

Case 2: time series

- Example



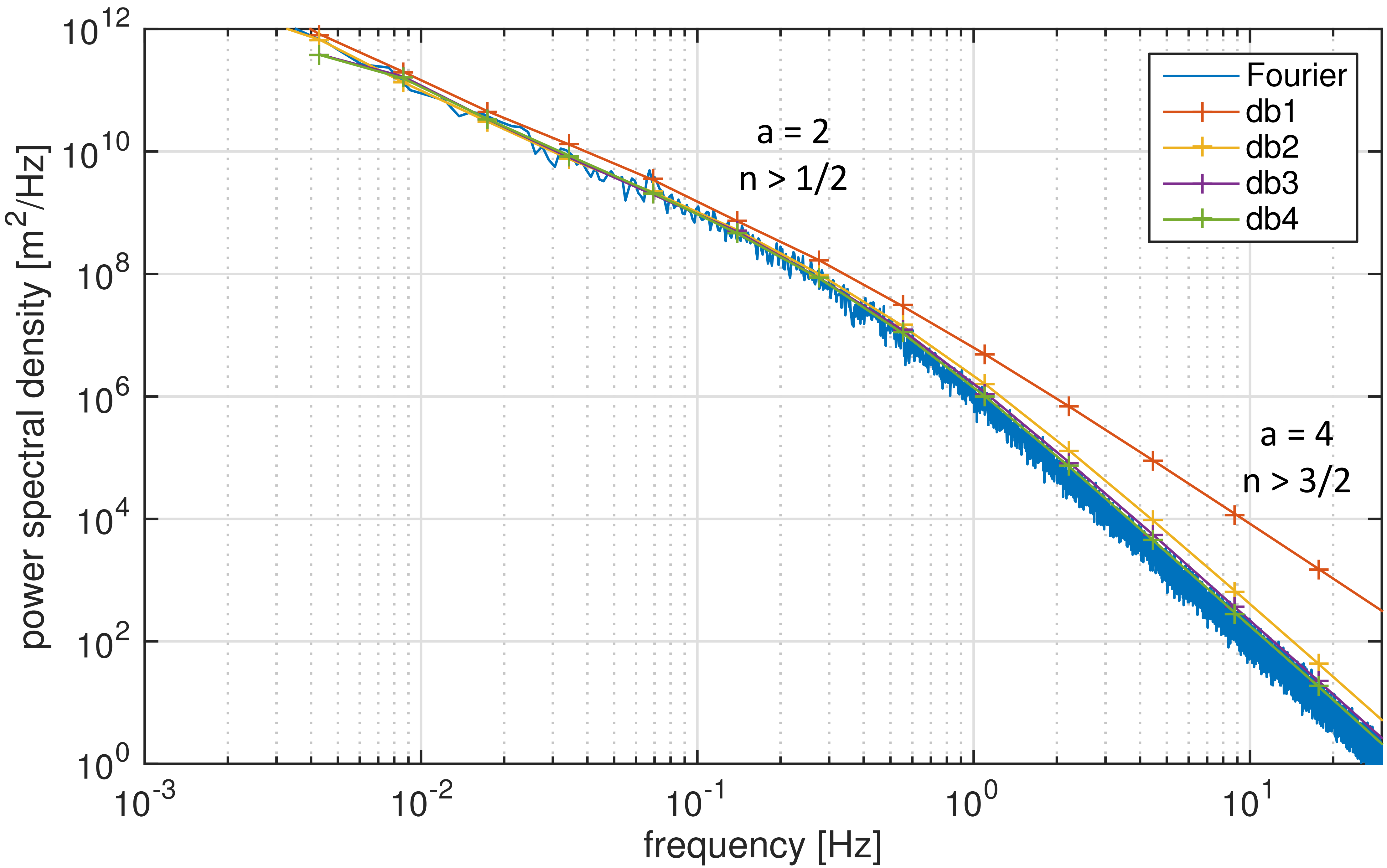
Why the order of the wavelet is important

If the mother wavelet does not have enough vanishing moments n

$$\int t^n \, \psi(t) \, \mathrm{d}t = 0$$

- The wavelet coefficients ($|\tilde{y}_{j,k}|^2$) are not independent anymore
- The estimate of the spectral index will be biased

Why the order of the wavelet is important



For estimating a power law $S(f) \sim f^{-a}$, the number of vanishing moments must be $n > (a-1)/2$

Conclusion

(not a fake one)

Conclusions (1/2)

1. almost any gentle curve will look like a line on a log-log plot
2. plotting scalograms, local intermittency measures, power laws, etc. without proper statistical testing is nonsense.

Conclusions (2/2)

- For sampled data
  - Use maximum likelihood estimator
  - Proper estimation of upper and lower (ymin) bounds is crucial
- For time series
  - Use discrete wavelet transform
  - Ensure that the order of the wavelet is high enough (stay away from Haar)
  - Power law should cover at least a decade to make sense.

- Not addressed here:
  - Use statistical tests (e.g. Kolmogorov-Smirnov) to compare power law model against other plausible models.
  - Or better, use a bayesian approach (computationally expensive)

Further reading

- P. Abry, P. Flandrin, M. S. Taqqu, and D. Veitch, *Self-similarity and long-range dependence through the wavelet lens*, Birkhäuser, 2003, 527–556.
- A. Clauset et al., *Power-law distributions in empirical data*, SIAM Review 51 (2009), 661-773
- Y. Virkar and A. Clauset, *Power-law distributions in binned empirical data*, Annals of Applied Statistics, 8 (2014), 89–119.